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Abstract

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FUNDAMENTAL SOLUTIONS AND THE CAUCHY PROBLEM FOR LINEAR PARABOLIC SYSTEMS WITH A BESSEL OPERATOR

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Fundamental solutions and the Cauchy problem for systems uniformly parabolic in the sense of Petrovskii have now been studied quite completely ([1,2]). Here analogous questions are solved for parabolic systems in which, along with ordinary derivatives, powers of the Bessel operator occur; this corresponds to systems expressed on a hyperplane. For systems of linear equations with constant coefficients in partial derivatives with Bessel operators, Ya. I. Zhitomirskii ([3]) proved the correct solvability of the Cauchy problem in the class of generalized functions. To obtain explicit formulas for the solution he made essential use of generalized shift operators corresponding to the Bessel operator, which were investigated by Ya. B. Levitan ([4]).

In the works of I. A. Kipriyanov and V. I. Kononenko ([5,6]), boundary-value problems were studied and fundamental solutions were constructed for elliptic equations containing powers of the Bessel operator.

1. Definition of the Green matrix. Consider the Cauchy problem for the system of differential equations

$$LU \equiv \frac{\partial U}{\partial t} - \sum_{|k|+2j\le 2b} (-i)^{|k|} A_{kj}(t,x) D^k_{x'} (-B_{x_{n+1}})^j U(t,x) = f(t,x), \qquad (1)$$

$$U\big|_{t=+t_0} = \varphi(x), \qquad \partial U/\partial x_{n+1}\big|_{x_{n+1}=0} = 0, \qquad (2)$$

where $B_{x_{n+1}}$ is the Bessel operator

$$B_{x_{n+1}} = \frac{\partial^2}{\partial x^2_{n+1}} + \frac{2\nu+1}{x_{n+1}}\frac{\partial}{\partial x_{n+1}}, \qquad \nu > -1/2;$$

$$(t,x) \in [t_0,T] \times E^+_{n+1} = \Pi^+;$$

E^+_{n+1} is the Euclidean space of points

$$(x_1, \dots, x_n, x_{n+1}) = (x', x_{n+1}), \qquad x_{n+1} \geq 0.$$

Definition 1. System (1) is called B-parabolic if, for every $\sigma \in E^+_{n+1}$ and all $(t, x) \in \Pi^+$, the roots of the equation

$$\det \left\| \sum_{|k|+2j=2b} A_{kj}(t,x) {\sigma'}^k \sigma_{n+1}^{2j} - \lambda E \right\| = 0$$

satisfy the inequality

$$\operatorname{Re} \lambda(t, x, \sigma) \leq -\delta(\sigma_1^2 + \dots + \sigma_{n+1}^2)^b,$$

where δ is a positive constant.

Definition 2. We shall call the Green matrix of the B-parabolic system

$$\frac{\partial U}{\partial t} = \sum_{|k|+2j \leq 2b} (-i)^{|k|} A_{kj}(t) D^k_{x'} (-B_{x_{n+1}})^{jU} \tag{3}$$

a matrix $G(t, \tau, x, \xi)$ such that the solution of the Cauchy problem (3), (2) for any initial smooth finite function $\varphi(x)$ is represented by the integral

$$U(t, x) = \int_{E^+_{n+1}} G(t, t_0, x, \xi) \varphi(\xi) \xi_{n+1}^{2\nu+1} \, d\xi.$$

If one seeks the solution of problem (3), (2) by means of the Fourier–Bessel transform

$$U(t, x) = \frac{1}{(2\pi)^n 2^{2\nu} \Gamma^2(\nu+1)} \int_{E^+_{n+1}} e^{i\sigma' \cdot x'} U(t, \sigma) j_\nu(x_{n+1}\sigma_{n+1}) \sigma_{n+1}^{2\nu+1} \, d\sigma,$$

then for $G(t, \tau, x, \xi)$ we obtain the formula

$$G(t, \tau, x, \xi) = T^{\xi_{n+1}}_{x_{n+1}} G(t, \tau, x' - \xi', x_{n+1}) = C_\nu \int_{E^+_{n+1}} e^{i\sigma' \cdot (x' - \xi')} Q(t, \tau, \sigma) j_\nu \times$$

$$\times (x_{n+1}\sigma_{n+1}) j_\nu(\xi_{n+1}\sigma_{n+1}) \sigma_{n+1}^{2\nu+1} \, d\sigma,$$

where

$$G(t,\tau,x) = C_\nu \int_{E^+_{n+1}} e^{i\sigma'\cdot x'} Q(t,\tau,\sigma) j_\nu(x_{n+1}\sigma_{n+1})\sigma_{n+1}^{2\nu+1}\, d\sigma;$$

$$C_\nu = \frac{1}{(2\pi)^n 2^{2\nu}\Gamma^2(\nu+1)};$$

$T^{\xi_{n+1}}_{x_{n+1}}$ is the generalized shift operator

$$T^{\xi_{n+1}}_{x_{n+1}} f(x_{n+1}) = \frac{\Gamma(\nu+1)}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_0^\pi f\left(\sqrt{x_{n+1}^2 + \xi_{n+1}^2 - 2x_{n+1}\xi_{n+1}\cos\alpha}\right)(\sin^{2\nu})\alpha\, d\alpha;$$

j_ν is the normalized Bessel function; $Q(t,\tau,\sigma)$ is the normal matrix of the system

$$\frac{dV}{dt} = \sum_{|k|+2j\le 2b} A_{kj}(t)\sigma'^k \sigma_{n+1}^{2j} V.$$

Theorem 1. The matrix $G(t,\tau,x,\xi)$, regarded as a function of the complex arguments $(x'-\xi')(t-\tau)^{-1/2b}$, $x_{n+1}(t-\tau)^{-1/2b}$, $\xi_{n+1}(t-\tau)^{-1/2b}$, is an entire function of order of growth $q = 2b/(2b-1)$, and for real arguments the estimates hold

$$\left|D^k_{x'} B^j_{x_{n+1}} G(t,\tau,x,\xi)\right| \le C_{kj}(t-\tau)^{-[n+2j+|k|+2(\nu+1)]/2b} \times$$

$$\times \exp\left\{-c\sum_{s=1}^n \left(\frac{x_s-\xi_s}{(t-\tau)^{1/2b}}\right)^q\right\} T^{\xi_{n+1}}_{x_{n+1}}\left\{\exp\left[-c\left(\frac{x_{n+1}}{(t-\tau)^{1/2b}}\right)^q\right]\right\}. \quad (4)$$

The positive constants C_{kj}, c depend on $\sup|A_{kj}(t)|$, the numbers $n, 2b, \delta, T$, and the character of continuity of $A_{kj}(t)$ with $|k|+2j = 2b$.

2. On the existence of a fundamental solution and the correct solvability of the Cauchy problem

First we define the Banach space in which problem (1), (2) is considered.

Definition 3. A function $U(t,x)$, whose derivatives appearing in system (1) are continuous in $\Pi^+_{(t_0,T]}$ for $t > t_0$ (i.e. $U \in C^{(2b,1)}_{x,t}$), belongs to the space $C^{(2b,\omega_3)}_{\omega_1,\omega_2}(\Pi^+_{(t_0,T]})$, if the norm is finite

$$|U|^{2b,\omega_3}_{\omega_1,\omega_2} = |U|_{2b,\omega_1} + [U]^{\omega_3}_{2b,\omega_2},$$

where

$$|U|_{2b,\omega_1} = \sum_{1\le|k|+2j\le 2b} \sup_{\Pi^+} \left\{ \frac{(t-t_0)^{|k|/2b}}{\omega_1[(t-t_0)^{1/2b}]} \left| D_x^k B_{x_{n+1}}^j U \right| \right\} + \sup_{\Pi^+} |U|;$$

$$[U]_{2b,\omega_2}^{\omega_3} = \sum_{|k|+2j=2b} \sup_{\Pi^+} \left\{ \frac{t-t_0}{\omega_2[(t-t_0)^{1/2b}]} \frac{\left| D^k B^{jU}(t,x) - D^k B^{jU}(t,\xi) \right|}{\omega_3(|x-\xi|)} \right\};$$

$\omega_i(h)$ are functions possessing the properties of a modulus of continuity. If in $C_{\omega_1,\omega_2}^{(2b,\omega_3)}$ one of the functions ω_i is identically constant, then in the corresponding place we shall write unity.

Theorem 2. *Suppose that the coefficients $A_{kj}(t,x)$ in Π^+ of the B-parabolic system (1) with $|k|+2j = 2b$ belong to the class H^2 with modulus of continuity $\omega(h)$ and are continuous in t uniformly with respect to x, while for $|k|+2j < 2b$ they belong to the class H^1 with modulus ω. Then the homogeneous system (1) has a fundamental matrix of solutions $Z(t,\tau,x,\xi)$, the derivatives $D_x^k B_{x_{n+1}}^{jZ}(t,\tau,x,\xi)$ with $|k|+2j \le 2b$ of which satisfy inequalities (4), and the following estimates hold*

$$\begin{aligned} \left| \Delta_h D_x^k B_{x_{n+1}}^j Z(t,\tau,x,\xi) \right| &\le \widetilde{C}_{kj} (t-\tau)^{-[n+2j+|k|+2(\nu+1)]/2b} [F(|h|) + \widetilde{F}(|h|)] \\ &\times \left\{ F[(t-t_0)^{1/2b}] + \widetilde{F}[(t-t_0)^{1/2b}] \right\}^{-1} \\ &\times \left\{ \exp\left[-c \sum_{s=1}^n \left(\frac{|x_s + h_s - \xi_s|}{(t-\tau)^{1/2b}} \right)^q \right] T_{x_{n+1}}^{\xi_{n+1}} \exp\left[-c \left(\frac{x_{n+1} + h_{n+1}}{(t-\tau)^{1/2b}} \right)^q \right] \right. \\ &\left. + \exp\left[-c \sum_{s=1}^n \left(\frac{x_s - \xi_s}{(t-\tau)^{1/2b}} \right)^q \right] T_{x_{n+1}}^{\xi_{n+1}} \exp\left[-c \left(\frac{x_{n+1}}{(t-\tau)^{1/2b}} \right)^q \right] \right\}, \end{aligned}$$

where $\Delta_h f(x) = f(x+h) - f(x)$; the constants $\widetilde{C}_{kj}, c$ depend on the same quantities as in Theorem 1, and also on $\sup(\omega, \tilde{\omega}, F, \widetilde{F})$,

$$F(a) = \int_0^a \frac{\omega(z)}{z}\, dz, \qquad \widetilde{F}(a) = \int_0^a \frac{\tilde{\omega}(z)}{z}\, dz, \qquad 0 < a < \infty.$$

The unique solution of the Cauchy problem (1), (2) in the class $C_{x,t}^{(2b,1)}(\Pi_{(t_0,T]}^+)$ for any continuous bounded function $\varphi(x)$ and $f \in C_{\omega,1}^{(0,\omega)}(\Pi^+)$ is determined by the formula

$$U(t,x) = \int_{E^+_{n+1}} Z(t,t_0,x,\xi)\varphi(\xi)\xi^{2\nu+1}_{n+1}\,d\xi + \int_0^t d\tau \int_{E^+_{n+1}} Z(t,\tau,x,\xi)f(\tau,\xi)\xi^{2\nu+1}_{n+1}\,d\xi,$$

and it belongs to the space $C^{(2b,F)}_{1,F^{-1}}(\Pi_{(t_0,T]})$.

Denote by $\mathcal{K}$ the operator of the Cauchy problem (1), (2): $KU = (f,\varphi)$. Define the Banach spaces

$$H \equiv H^{(\omega,F)}_{\omega,F^{-1}}, \qquad \mathcal{E} = C^{(0,\omega)}_{\omega,\omega}(\Pi^+_{(t_0,T]}) \times C^{(0,\omega)}(E^+_{n+1})$$

with norms

$$|U|_H = |U(t_0,x)|^{\omega}_0 + |LU|^{(0,\omega)}_{\omega,\omega} + |U|^{(2b,F)}_{\omega,F^{-1}} + \sup_{\Pi^+}\left\{\frac{t-t_0}{\omega[(t-t_0)^{1/2b}]}\left|\frac{\partial U}{\partial t}\right|\right\},$$

$$|(f,\varphi)|_E = |f|^{0,\omega}_{\omega,\omega} + |\varphi|^{\omega}_0.$$

Obviously, $|KU|_E \le |U|_H$.

Theorem 3 (on homeomorphism). *If the coefficients of the B-parabolic system (1) belong only to the class* H^1 *with modulus of continuity* ω*, and* (f,φ) *are arbitrary functions from the space* $\mathcal{E}$*, then in the space* H *the problem (1), (2) has a unique solution. The solution of the Cauchy problem (1),*

(2) from the space $H^{(\omega,\theta)}_{\omega,F^{-1}}$ belongs to the space H, and the estimate holds

$$|U|_H \le C|(f,\varphi)|_{\mathcal{E}},$$

where C depends on the same quantities as in Theorem 2.

Thus, the operator $\mathcal{K}$ is defined on the Banach space H and acts from H into the Banach space $\mathcal{E}$ as a bounded operator; it has a bounded inverse acting from the space $\mathcal{E}$ into H.

The results obtained are also valid in the corresponding Dini spaces of exponentially increasing functions ([2]).

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Note: Figure translations are in progress. See original paper for figures.