ON INTEGRAL-DIFFERENCE WIENER-HOPF EQUATIONS

MATHEMATICS

1968

SovietRxiv

Abstract

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UDC 517.948.3/5:513.88

MATHEMATICS

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ON INTEGRAL-DIFFERENCE WIENER-HOPF EQUATIONS

(Presented by Academician N. I. Muskhelishvili, 6 III 1968)

1. In the present note we study integral-difference equations of the form

$$\sum_{j=-\infty}^{\infty} a_j \varphi(t-\delta_j) + \int_0^{\infty} k(t-s)\varphi(s)\,ds = f(t) \qquad (0 \le t < \infty), \tag{1}$$

where a_j $(j = 0, \pm 1, ...)$ are prescribed complex numbers; δ_j $(j = 0, \pm 1, ...)$ are prescribed real numbers, $k(t)$ $(-\infty < t < \infty)$ and $f(t)$ $(0 \le t < \infty)$ are prescribed functions, and $\varphi(t)$ $(0 \le t < \infty)$ is the unknown function. In addition, in (1) it is assumed that, for $\delta_j > 0$, the function $\varphi(t-\delta_j)$ is set equal to zero on the interval $0 \le t \le \delta_j$.

Equation (1) is a generalization both of the Wiener-Hopf integral equation and of its discrete analogue. If one puts $a_j = 0$ $(j \ne 0)$ and $\delta_0 = 0$, then equation (1) coincides with the Wiener-Hopf integral equation; if $k(t) \equiv 0$ and $\delta_j = j$ $(j = 0, \pm 1, ...)$, then, with respect to the vector-function $\varphi(t) = \{\varphi_1(t), \varphi_2(t), ...\}$, where $\varphi_j(t) = \varphi(t+j-1)$ $(0 \le t \le 1)$, equation (1) is the discrete Wiener-Hopf equation with matrix $\|a_{j-k}\|_{i,k=1}^{\infty}$.

Below equation (1) will be considered in the space $L_p(0,\infty)$ $(1 \le p \le \infty)$ under the following restrictions:

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty, \qquad \int_{-\infty}^{\infty} |k(t)|\,dt < \infty. \tag{2}$$

The results obtained in this note generalize the results of M. G. Krein ([1]) on Wiener-Hopf equations.

Let us also note that, under the restrictions (2), equation (1) can be written in the form

$$\int_{-\infty}^{t} \varphi(t-s)\,d\omega(s) = f(t) \qquad (0 \le t < \infty),$$

where $\omega(t)$ $(-\infty < t < \infty)$ is a function of bounded variation without singular component (see ([2])).

2. Denote by $\mathfrak{A}$ the set of all operators W acting in the space $L_p(0,\infty)$ by the formula

$$(W\varphi)(t) = \sum_{j=-\infty}^{\infty} a_j \varphi(t-\delta_j) + \int_0^{\infty} k(t-s)\varphi(s)\,ds, \tag{3}$$

where the numbers a_j and the function $k(t)$ satisfy conditions (2).

To the operator $W \in \mathfrak{A}$, defined by equality (3), we associate the function

$$\mathcal{W}(\lambda) = \sum_{j=-\infty}^{\infty} a_j e^{i\delta_j\lambda} + \int_{-\infty}^{\infty} k(t)e^{i\lambda t}\,dt \qquad (-\infty < \lambda < \infty). \tag{4}$$

which we shall call the **symbol** of the operator W. We shall call the symbol $\mathcal{W}(\lambda)$ **nondegenerate** if

$$\inf |\mathcal{W}(\lambda)| > 0.$$

It is easy to see that every operator $W \in \widetilde{\mathfrak{A}}$ is a linear bounded operator in any of the spaces $L_p(0,\infty)$ $(1 \le p \le \infty)$.

Theorem 1. *In order that an operator $W \in \widetilde{\mathfrak{A}}$ be invertible in the space $L_p(0,\infty)$ $(1 \le p \le \infty)$ at least on one side, it is necessary and sufficient that its symbol $\mathcal{W}(\lambda)$ be nondegenerate.*

*If the latter condition is not satisfied, then the operator W is neither a Φ- nor a $\Phi_\pm$-operator.**

3. Denote by $\mathfrak{A}$ the algebra of all functions $\mathcal{W}(\lambda)$ of the form (4) satisfying condition (2).

By $\mathfrak{A}_+(\mathfrak{A}_-)$ we denote the subalgebra of the algebra $\mathfrak{A}$ consisting of all functions of the form (4) for which $\delta_j \ge 0$ and $k(t) = 0$ for $t \le 0$ ($\delta_j \le 0$ and $k(t) = 0$ for $t \ge 0$).

The algebra $\mathfrak{A}$ is the direct sum of the subalgebra $\mathfrak{B}$ of almost periodic functions of the form $\sum a_j \exp(i\delta_j\lambda)$ $(\sum |a_j| < \infty)$ and the subalgebra $\mathfrak{C}$ of Fourier transforms of functions from $L_1(-\infty,\infty)$. The subalgebra $\mathfrak{C}$ is an ideal of the algebra $\mathfrak{A}$.

If the function $\mathcal{W}(\lambda) = a(\lambda) + \mathcal{K}(\lambda)$ $(a(\lambda) \in \mathfrak{B},\ \mathcal{K}(\lambda) \in \mathfrak{C})$ is nondegenerate, then its almost periodic component $a(\lambda)$ is also nondegenerate. With every nondegenerate function $\mathcal{W}(\lambda) \in \mathfrak{A}$ are associated two numbers $\nu(\mathcal{W})$ and $n(\mathcal{W})$, defined by the equalities

$$\nu\mathcal{W} = \lim_{l\to\infty} \frac{1}{2l}\,[\arg a(\lambda)]_{-l}^{l}, \qquad n(\mathcal{W}) = \frac{1}{2\pi}\,[\arg(1 + a^{-1}(\lambda)\mathcal{K}(\lambda))]_{-\infty}^{\infty}.$$

The limit in the first equality exists by virtue of a known property of almost periodic functions (see (4)). The real number $\nu(\mathcal{W})$ and the integer $n(\mathcal{W})$ will be called the **indices** of the function $\mathcal{W}(\lambda)$.

Theorem 2. *Every nondegenerate function $\mathcal{W}(\lambda) \in \mathfrak{A}$ admits a factorization of the form*

$$\mathcal{W}(\lambda) = \mathcal{W}_-(\lambda) e^{i\lambda\nu} \left(\frac{\lambda - i}{\lambda + i}\right)^n \mathcal{W}_+(\lambda) \qquad (-\infty < \lambda < \infty), \tag{5}$$

where $\mathcal{W}_\pm(\lambda), \mathcal{W}_\pm^{-1}(\lambda) \in \mathfrak{A}_\pm$; $\nu = \nu(\mathcal{W})$ and $n = n(\mathcal{W})$.

This theorem plays an important role in the proof of the sufficiency of the conditions of Theorem 1, and also in the proof of the following Theorems 3-6. The proof of Theorem 2 is based on a theorem of G. E. Shilov (see (2)) on analytic functions of elements of a commutative Banach algebra.

4. For the formulation of the subsequent theorems we shall need the operator $V^{(n)}$ $(n = 0, \pm 1, \ldots)$, whose symbol is equal to $(\lambda - i)^n / (\lambda + i)^n$. It is easy to see that $V^{(n)} = V^n$ $(n = 1, 2, \ldots)$ and $V^{(n)} = (V^{-1})^{-n}$ $(n = -1, -2, \ldots)$, where V and V^{-1} are the operators defined by the equalities

$$(V\varphi)(t) = \varphi(t) - 2\int_0^t e^{s-t}\varphi(s)\,ds, \qquad (V^{-1}\varphi)(t) = \varphi(t) - 2\int_0^\infty e^{t-s}\varphi(s)\,ds.$$

Let us also note that if an operator $W (\in \widetilde{\mathfrak{A}})$ is such that both functions $\mathcal{W}^{\pm 1}(\lambda) \in \mathfrak{A}_+$ $(\mathcal{W}^{\pm 1}(\lambda) \in \mathfrak{A}_-)$, then it is invertible, with $W^{-1} \in \widetilde{\mathfrak{A}}$, and the symbol of the operator W^{-1} is equal to $1/\mathcal{W}(\lambda)$.

In all subsequent theorems we shall assume that $W (\in \widetilde{\mathfrak{A}})$ is an operator with nondegenerate symbol $\mathcal{W}(\lambda)$ and that equality (5) gives a factorization of the symbol $\mathcal{W}(\lambda)$. For brevity the numbers $\nu(\mathcal{W})$ and $n(\mathcal{W})$ will be denoted by ν and n. By $W_\pm$ is denoted the invertible operator with symbol $\mathcal{W}_\pm(\lambda)$.

Theorem 3. *If $\nu > 0$, then the operator W is left-invertible. In order that the equation $W\varphi = g$ be solvable, it is necessary and sufficient that:*

* For the definition of Φ- and $\Phi_\pm$-operators see (3).

a) for $n \geqslant 0$ the function $W_-^{-1} g$ vanishes on the interval $[0, \nu]$ and the condition

$$\int_0^\infty (W_-^{-1} g)(t) t^k e^{-t}\,dt = 0 \quad (k = 0, 1, \ldots, n-1); \tag{6}$$

is satisfied;

b) for $n < 0$ the function $V^{-n} W_-^{-1} g$ coincides on the interval $[0, \nu]$ with a function of the form

$$\sum_{j=0}^{-n-1} c_j t^j e^{-t},$$

where the c_j are complex numbers.

Theorem 4. If $\nu < 0$, then the operator W is right-invertible. Every solution $\varphi \in L_p(0, \infty)$ of the equation $W\varphi = 0$:

a) for $n \geqslant 0$ has the form

$$\varphi = W_+^{-1} V^{-n} g,$$

where g is an arbitrary function from $L_p(0, \infty)$ satisfying the conditions

$$g(t) = 0 \ (t > -\nu); \qquad \int_0^\infty g(t) t^j e^{-t}\, dt = 0 \quad (j = 0, 1, \dots, n-1);$$

b) for $n < 0$ has the form

$$\varphi = W_+^{-1} \left(g(t) + \sum_{j=0}^{-n-1} c_j t^j e^{-t} \right),$$

where g is an arbitrary function from L_p, vanishing on the interval $[-\nu, \infty]$, and the c_j are arbitrary complex numbers.

Theorem 5. If $\nu = 0$, then:

a) for $n = 0$ the operator W is invertible;

b) for $n > 0$ the operator W is left-invertible; the equation $W\varphi = g$ is solvable if and only if condition (6) is fulfilled;

c) for $n < 0$ the operator W is right-invertible and the general solution of the equation $W\varphi = 0$ is given by the equality

$$\varphi = W_+^{-1} \left(\sum_{j=0}^{-n-1} c_j t^j e^{-t} \right),$$

where the c_j are arbitrary complex numbers.

5. Denote by U_ν the operator with symbol $e^{i\nu\lambda}$, i.e., the operator defined by the equalities $(U_\nu f)(t) = f(t-\nu)$ for $\max(\nu, 0) < t < \infty$, and $(U_\nu f)(t) = 0$ for $0 \leqslant t \leqslant \max(\nu, 0)$.

It is easy to see that if the numbers ν and n are positive, then the operator $U_\nu V^{(n)}$ is only left-invertible, while the operator $U_{-\nu}V^{(-n)}$ is only right-invertible. In the case $\nu > 0$ and $n < 0$ the equality $U_{-\nu}V^{(-n)}V^{(n)}U_\nu = I - U_{-\nu}P_{-n}U_\nu$ holds, where $P_{-n} = I - V^{(-n)}V^{(n)}$ is a finite-dimensional projector. It is readily proved that the operator $I - U_{-\nu}P_{-n}U_\nu$ is invertible in every L_p $(1 \leqslant p \leqslant \infty)$.

The operator $W^{(-1)}$, inverse to the operator W from the corresponding side, is defined by the equalities

$$W^{(-1)} = W_+^{-1}V^{(-n)}U_{-\nu}W_-^{-1} \qquad (\nu \geqslant 0,\ n \geqslant 0);$$

$$W^{(-1)} = W_+^{-1}(I - U_{-\nu}P_{-n}U_\nu)^{-1}U_{-\nu}V^{(-n)}W_-^{-1} \qquad (\nu \geqslant 0,\ n \leqslant 0);$$

$$W^{(-1)} = W_+^{-1}V^{(-n)}U_{-\nu}W_-^{-1} \qquad (\nu \leqslant 0,\ n \leqslant 0);$$

$$W^{(-1)} = W_+^{-1}V^{(-n)}U_{-\nu}(I - U_\nu P_n U_{-\nu})^{-1}W_-^{-1} \qquad (\nu \leqslant 0,\ n \geqslant 0).$$

In the first two cases the operator $W^{(-1)}$ is a left inverse to the operator W, and in the last two cases the operator $W^{(-1)}$ is a right inverse to W. If at least one of the indices ν, n is nonzero, then the operator $W^{(-1)}$ is an inverse to W only from one side.

6. The restrictions on the symbol $\mathcal{W}(\lambda)$ can in some cases be weakened. For example, for the Hilbert space $L_2(0,\infty)$ the results listed above extend to the case of an operator W whose symbol $\mathcal{W}(\lambda) = a(\lambda) + \mathcal{K}(\lambda)$ is such that $a(\lambda)$ is an arbitrary almost periodic function, and $\mathcal{K}(\lambda)$ $(-\infty \le \lambda \le \infty)$ is an arbitrary continuous function.

The results listed above extend to the corresponding adjoint equations and to their transposes.

7. Denote by P_τ $(0 < \tau < \infty)$ the projector defined in the space $L_p(0,\infty)$ $(1 \le p < \infty)$ by the equality

$$(P_\tau\varphi)(t) = \begin{cases} \varphi(t), & t \le \tau, \\ 0, & t > \tau. \end{cases}$$

If the operator $P_\tau W P_\tau$ is invertible in the subspace $P_\tau L_p(0,\infty)$, then by $(P_\tau W P_\tau)^{-1}$ we shall denote the operator equal to the inverse of the operator $P_\tau W P_\tau$ on the subspace $P_\tau L_p(0,\infty)$ and equal to zero on the subspace $(I - P_\tau)L_p(0,\infty)$. If the operator $P_\tau W P_\tau$ is not invertible, then we put $\|(P_\tau W P_\tau)^{-1}\| = \infty$.

Theorem 6. Let $W \in \widetilde{\mathfrak{A}}$. If

$$\lim_{\tau\to\infty} \|(P_\tau W P_\tau)^{-1}\| < \infty,$$

then the symbol of the operator W is nondegenerate and $v(\mathcal{W}) = n(\mathcal{W}) = 0$.

When the last conditions are fulfilled, the equation

$$\sum_{t-\tau\le\delta_j<t} a_j\varphi(t-\delta_j) + \int_0^\tau k(t-s)\varphi(s)\,ds = f(t) \qquad (0 \le t \le \tau),$$

beginning with some τ, has a unique solution $\varphi_\tau(t) \in P_\tau L_p(0,\tau)$, and the functions

$$\widetilde{\varphi}_\tau(t) = \begin{cases} \varphi_\tau(t), & t \le \tau, \\ 0, & t > \tau. \end{cases}$$

converge as $\tau \to \infty$, in the norm of the space $L_p(0,\infty)$, to the solution of equation (1), whatever the function $f(t) \in L_p(0,\infty)$.

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Received
1 III 1968

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Note: Figure translations are in progress. See original paper for figures.