ON THE QUESTION OF THE STRUCTURE OF A NEIGHBORHOOD OF A HOMOCLINIC TUBE OF AN INVARIANT TORUS

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Abstract

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MATHEMATICS

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ON THE QUESTION OF THE STRUCTURE OF A NEIGHBORHOOD OF A HOMOCLINIC TUBE OF AN INVARIANT TORUS

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1. Let T be a smooth one-to-one mapping defined in some domain G of the Euclidean space R^{l+m+n}. Suppose that T has an invariant torus τ of dimension l and that in a neighborhood of τ T can be written in the form

$$\bar{x} = A(\theta)(x + f(x,y,\theta)), \quad \bar{y} = B(\theta)(y + g(x,y,\theta)), \quad \bar{\theta} = \Psi(\theta), \qquad (1)$$

where $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_n)$, $\theta = (\theta_1, \dots, \theta_l)$; $A, B, f, g, \Psi \in C^1$ and are periodic in θ with period $1 = (1, \dots, 1)$, $f = g = f_x = f_y = f_\theta = g = g_x = g_y = g_\theta = 0$ for $x = y = 0$.

Suppose that

$$\max \|A(\theta)\| < 1, \quad \max \|B^{-1}(\theta)\| < 1. \qquad (2)$$

Conditions (2) mean that the torus τ is saddle. It can be shown that under the conditions

$$\bar{p} = \max \|A(\theta)\| \max \|\Psi_\theta^{-1}\| < 1, \quad \bar{p} = \max \|B^{-1}(\theta)\| \max \|\Psi_\theta\| < 1 \qquad (3)$$

the stable and unstable manifolds of τ, which we denote by $\mathfrak{M}^+$ and $\mathfrak{M}^-$, in a sufficiently small neighborhood of τ are written in the form $y = R^+(x,\theta)$, $x = R^-(y,\theta)$, where $R^+, R^- \in C^1$ and $R^+(0,\theta) = R^-(0,\theta) = R^+_x(0,\theta) = R^+_\theta(0,\theta) = R^-_y(0,\theta) = R^-_\theta(0,\theta) = 0$. Everywhere below it is assumed that conditions (3) are satisfied and $f(0,y,\theta) = g(x,0,\theta) = 0$.

Suppose that $\mathfrak{M}^+$ and $\mathfrak{M}^-$ intersect along an l-dimensional manifold γ of torus type, which we shall call a homoclinic torus.* With respect to this intersection suppose that it is rough, i.e., for an arbitrary point $M \in \gamma$,

$$\dim(W^+_M \cap W^-_M) = l, \qquad (4)$$

where W_M^+ and W_M^- denote the tangent spaces to $\mathfrak{M}^+$ and $\mathfrak{M}^-$ at the point M.

The presence of a homoclinic torus γ leads to the existence of a countable set of homoclinic tori $\gamma_i = T^i\gamma$, $i = 0, \pm 1, \pm 2, \ldots$. The sequence $\Gamma_0 = (\ldots, \gamma_{-1}, \gamma_0, \gamma_1, \ldots, \gamma_i, \ldots)$ will be called a homoclinic tube of the torus τ.

Let $\gamma^- = T^{-p_1}\gamma$ and $\gamma^+ = T^{p_2}\gamma$ be two homoclinic tori of the tube Γ_0, lying entirely in a sufficiently small neighborhood of τ. We shall suppose that their equations can be written in the form

$$\gamma^- : \ x = 0, \ y = \varphi^-(\theta); \quad \gamma^+ : \ x = \varphi^+(\theta), \ y = 0, \tag{5}$$

where φ^- and φ^+ are smooth functions, periodic in θ with period 1.

The aim of the paper is to determine the possible types of invariant tubes lying entirely in a neighborhood of the tube Γ_0.

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* The existence of an intersection of stable and unstable manifolds of saddle-type tori was considered by V. I. Arnold ([4]) in a problem illustrating the effect of instability of conservative systems with many degrees of freedom.

2. Consider the system

$$x_k = A_{kx}0 + \sum_{j=0}^{k} A_{k-j} f(x_j, y_j, \theta_j), \qquad y_k = B_{k-N} y_N - \sum_{j=k}^{N} B_{k-j} g(x_j, y_j, \theta_j),$$

$$\theta_k = \Psi_k(\theta_0), \tag{6}$$

where

$$A_{k-j} = A(\theta_{k-1}) \ldots A(\theta_j), \qquad B_{k-j} = B^{-1}(\theta_k) \ldots B^{-1}(\theta_{j-1}).$$

$$\Psi_k = \Psi(\ldots \underbrace{\Psi}_{k}(\theta_0))$$

and it is assumed that $A_0 f = 0, \ B_{k-N} g = 0, \ A_0 = E_m, \ B_0 = E_n$.

Lemma 1. There exists an r such that, for $\|x_0\| < r/2, \ \|y_N\| < r/2$, system (6) has a unique solution

$$(x_k(x_0, y_N, \theta_0), \ y_k(x_0, y_N, \theta_0), \ \theta_k = \Psi_k(\theta_0))_{k=0}^{k=N},$$

where $\|x_k\| < r, \ \|y_k\| < r$.

The proof follows from the contraction mapping method.

Lemma 2. The estimates

$$\|x_N\| + \|\partial x_N/\partial x_0\| + \|\partial x_N/\partial y_N\| < L(\max\|A(\theta)\|)^N,$$

$$\|y_0\| + \|\partial y_0/\partial x_0\| + \|\partial y_0/\partial y_N\| < L(\max\|B^{-1}(\theta)\|)^N, \tag{7}$$

$$\|\partial x_N(x_0, y_N, \Psi_N^{-1}(\theta_N))/\partial\theta_N\| < L\bar{\rho}^N, \qquad \|\partial y_0/\partial\theta_0\| < L\bar{\rho}^N,$$

hold, where L is some constant.

Let

$$U_0 = [M_0(x_0, y_0, \theta_0);\ \|x_0 - \varphi^+(\theta_0)\| \le \varepsilon_0,\ \|y_0\| \le \varepsilon_0,\ \theta_0 \in \tau]$$
$$U_1 = [M_1(x_1, y_1, \theta_1);\ \|x_1\| \le \varepsilon_1,\ \|y_1 - \varphi^-(\theta_1)\| \le \varepsilon_1,\ \theta_1 \in \tau].$$

Choose ε_0 so small that, for the neighborhood τ under consideration, $T_0^{iM} \cap U_0 = \emptyset$, where $i \ne 0$ and $M_0 \in U_0$. We impose an analogous restriction on ε_1.

Lemma 3. For any k greater than some $\bar{k}_1$, the mapping T_0^{kU} into U_1 is defined and is written in the form

$$\theta_1 = \Psi_k(\theta_0),$$

$$\xi_1 = x_k(\xi_0 + \varphi^+(\theta_0),\ \eta_1 + \varphi^-(\Psi_k^{-1}(\theta_0)),\ \theta_0), \tag{8}$$

$$\eta_0 = y_0(\xi_0 + \varphi^+(\theta_0),\ \eta_1 + \varphi^-(\Psi_k^{-1}(\theta_0)),\ \theta_0), \tag{9}$$

where

$$\xi_0 = x_0 - \varphi^+(\theta_0), \quad \eta_0 = y_0, \quad \xi_1 = x_1, \quad \eta_1 = y_1 - \varphi^-(\theta_1).$$

Denote by σ_k^0 the domain of definition of $T^k : U_0 \to U_1$. From the form of T^k it follows that σ_k^0 fibers into the surfaces (9), which tend smoothly to $y_0 = 0$ as $k \to \infty$. Note that

$$\sigma_{k_1}^0 \cap \sigma_{k_2}^0 = \emptyset, \qquad \sigma_{k_1}^1 \cap \sigma_{k_2}^1 = \emptyset, \quad \text{where } \sigma_k^1 = T^k\sigma_k^0 \text{ and } k_1 \ne k_2.$$

The mapping $T^{p_1+p_2}$ in a neighborhood of γ^-, which may be taken to be U_1, in the variables ξ, η, θ is written in the form

$$\bar{\xi}_0 = F(\xi_1, \eta_1, \theta_1), \qquad \bar{\eta}_0 = G(\xi_1, \eta_1, \theta_1), \qquad \bar{\theta}_0 = \Phi(\xi_1, \eta_1, \theta_1). \tag{10}$$

By assumption, $\mathfrak{M}^+$ and $\mathfrak{M}^-$ intersect transversely, and the equations of γ^- and γ^+ are written in the form (5). For these conditions to hold it is sufficient that

$$\|G_{\eta_1}(0,0,\theta_1)\| \neq 0, \qquad \begin{Vmatrix} G_{\eta_1} & G_{\theta_1} \\ \Phi_{\eta_1} & \Phi_{\theta_1} \end{Vmatrix}_{\substack{\xi_1=0\\ \eta_1=0}} \neq 0. \tag{11}$$

When (11) is fulfilled, the equation of $\mathfrak{M}^-$ can be written in the form

$$\xi_0 = \varphi_0^0(\eta_0, \theta_0), \tag{12}$$

where φ_0^0 is a smooth function, defined for $\|y_0\| \leq \varepsilon_0'$, periodic in θ_0 with period 1, and

$$\max(\|\partial\varphi_0^0/\partial\eta_0\|,\ \|\partial\varphi_0^0/\partial\theta_0\|) < L_0.$$

Introduce the space H^0 of surfaces P^0 with norm c^0, whose equations are written in the form

$$\xi_0 = \varphi_0(\eta_0, \theta_0),$$

where the functions φ_0 are defined for $\|\eta_0\| \leq \varepsilon_0'/2$, periodic in θ_0 with period 1, and satisfy the conditions

$$\|\varphi_0(0,\theta_0)\| < \varepsilon_0/2, \qquad \|\varphi_0(\eta_0',\theta_0') - \varphi_0(\eta_0'',\theta_0'')\| < (L_0+1)(\|\eta_0' - \eta_0''\| + |\theta_0' - \theta_0''|).$$

Denote by H_k^0 the set of surfaces

$$P_k^0 = P^0 \cap \sigma_k^0.$$

Theorem 1. There exists a $\bar{k}_2 \geq \bar{k}_1$ such that for all $i, j \leq \bar{k}_2$:
1) $T^{j+P_1+P_2}\sigma_j^0 \cap \sigma_i^0 \neq \emptyset$;
2) the operators $\mathcal{T}_{ij}^+ : H_j^0 \to H_i^0$ are defined;
3) in some metric equivalent to C^0,

$$\rho(\mathcal{T}_{ij}^+\varphi_0', \mathcal{T}_{ij}^+\varphi_0'') < q\rho(\varphi_0', \varphi_0''),$$

where $q < 1$.

3. Let

$$\pi = (\dots, i_{-1}, i_0, i_1, \dots, i_p, \dots) \tag{13}$$

be a sequence infinite in both directions, composed of integers $i_p \geq \bar{k}_2$. To the sequence π we assign the sequence

$$\to H_{i_{p-1}}^0 \xrightarrow{\mathcal{T}_{i_p i_{p-1}}^+} H_{i_p}^0 \xrightarrow{\mathcal{T}_{i_{p+1} i_p}^+} H_{i_{p+1}}^0 \to \dots \tag{14}$$

Lemma 4 ([6]). Let there be given a sequence of complete metric spaces X_i with metric ρ_i, $i = 0, \pm1, \pm2, \dots$, and a sequence of operators A_i satisfying the following conditions:
1) $\sup \rho_i(x_i'', x_i') < D$;

2) $A_i(X_i) \subset X_{i+1}$;
3) $\rho_{i+1}(A_i x_i', A_i x_i'') < q\rho_i(x_i', x_i'')$, where $q < 1$.
Then there exists a unique sequence $(\dots, x_{-1}^*, x_0^*, \dots, x_i^*, \dots)$, $x_i^* \in X_i$, satisfying the conditions $x_{i+1}^* = A_i x_i^*$.

Applying this lemma to (14), we obtain that to the sequence π there corresponds a unique stable sequence of surfaces $\{P_{i_p}^\pi\}$, whose equations

$$\xi_0 = \varphi_{i_p}^\pi(\eta_0, \theta_0),$$

satisfy the conditions

$$T^{-i_0-p_1-p_2} P_{i_{p+1}}^\pi \subset P_{i_p}^\pi. \tag{15}$$

In an analogous way it is established that to any sequence π, where $i_p \geq \bar{k}_2'$, there corresponds a unique sequence of surfaces $\{Q_{i_p}^\pi\}$ stable in the negative direction, whose equations are

$$\eta_0 = \psi_{i_p}^\pi(\xi_0, \theta_0),$$

where $\psi_{i_p}^\pi$ are defined for $\|\xi_0\| \leq \varepsilon_0$, are periodic with respect to θ_0, their graphs lie in $\sigma_{i_p}^0$, and $\psi_{i_p}^\pi \to 0$ as $i_p \to \infty$, together with the Lipschitz constants.

For all $i_p \geq \bar{k} \geq \max(\bar{k}_2, \bar{k}_2')$, $P_{i_p}^\pi$ and $Q_{i_p}^\pi$ intersect in a unique l-dimensional torus $\gamma_{i_p}^\pi$. It follows from the construction that

$$\Gamma^0 = (\dots, \gamma_{i_p}^\pi, \dots, \gamma_{i_{p+1}}^\pi, \dots)$$

is an invariant tube.

As a result we arrive at the following theorem.

Theorem 2. The set of invariant tubes lying entirely in a sufficiently small neighborhood of Γ_0 and not asymptotic to τ is in one-to-one correspondence with the set of all sequences π, where $i_p \geq k^* \geq \bar{k}$.

Corollary. In any neighborhood of Γ_0 there is contained a countable set of periodic tori.

Concerning the existence of asymptotic tubes in a small neighborhood of Γ_0, the following holds.

Theorem 3. The set of invariant tubes asymptotic to τ only in the negative (positive) direction is in one-to-one correspondence with the set of all infinite sequences $\pi = (i_1, i_2, \dots, i_p, \dots)$, where $i_p \geq k^* \geq \bar{k}$.

The set of homoclinic tubes of the torus τ, excluding Γ_0, is in one-to-one correspondence with the set of all segments $(\nu_1, \dots, \nu_q)$, where $\nu_q \geq k^*$, $q \geq 1$.

Remark 1. In the case when T has a fixed point of saddle type, whose stable and unstable manifolds intersect transversely in a homoclinic point, Theorems

2 and 3 give a description of the structure of a neighborhood of the homoclinic trajectory. A somewhat different proof was earlier proposed by the author ([5],[6]) in the study of continuous dynamical systems.

Remark 2. Conditions (3) are automatically fulfilled when the motions on the torus are quasiperiodic. However, in the general case this is not so.

In particular, conditions (3) may fail when the mapping on the invariant torus has a Y-structure in the sense of D. V. Anosov ([1–3]).*

Remark 3. Although the paper carried out the study under the assumption that the invariant manifold of the mapping T is a torus, the results are also valid for any smooth manifold under analogous assumptions with respect to T. Examples of such invariant manifolds may be R^1, $R^1 \times S^1$, etc.

4. Consider a system of $m+n$ equations

$$dx_i/dt = f_i(x_1, \dots, x_{m+n}, t), \tag{16}$$

where the f_i are defined and continuous, together with the derivatives $\partial f_i/\partial x_j$, in the domain $G \times R^1$, where G is a bounded closed domain in R^{m+n}, and are quasiperiodic functions of t with frequency basis $\omega_1, \dots, \omega_k$. Let $x = 0$ be a solution, and let the system

$$\frac{d\xi_i}{dt} = \sum a_{ij}(t)\xi_j, \tag{17}$$

where $a_{ij} = \partial f_i(0,t)/\partial x_j$, have a fundamental matrix representable in the form

$$\begin{pmatrix} \Psi^+(t) & 0 \\ 0 & \Psi^-(t) \end{pmatrix}, \tag{18}$$

where Ψ^+ and Ψ^- are matrices of orders m and n, respectively, satisfying the conditions

$$\begin{aligned} \|\Psi^+(t)(\Psi^+(\tau))^{-1}\| &< \bar{k}e^{-\lambda(t-\tau)}, \qquad t-\tau \geq 0, \\ \|\Psi^-(t)(\Psi^-(\tau))^{-1}\| &< \bar{k}e^{\lambda(t-\tau)}, \qquad t-\tau \leq 0, \end{aligned} \tag{19}$$

where $\bar{k}$ and λ are some positive constants. Then the solution $x = 0$ is a solution of saddle type and has a stable integral manifold $\mathfrak{M}^+_{m+1}$ of dimension $m+1$ and an unstable $\mathfrak{M}^-_{n+1}$ of dimension $n+1$. Suppose that there exists a solution $x = \varphi(t)$, doubly asymptotic to $x = 0$, and that $\mathfrak{M}^+_{m+1}$ and $\mathfrak{M}^-_{n+1}$ intersect along $x = \varphi(t)$ transversally.

Theorem 4. System (16) has a countable set of quasiperiodic motions of saddle type with frequency basis $\omega_1, \dots, \omega_k$.

The proof is based on the following

Lemma 5. The stable and unstable manifolds $\mathfrak{M}^+_{m+k}$ and $\mathfrak{M}^-_{n-k}$ of the torus $x = 0$ of the system

$$dx_i/dt = g_i(x_i, \dots, x_{m+n}, \theta_1, \dots, \theta_k), \qquad d\theta_j/dt = \omega_j, \tag{18'}$$

where $g_i(x, \omega_1, t, \dots, \omega_k t) = f(x, t)$, intersect transversally along a k-dimensional homoclinic tube.

Applying the results obtained above to system (18′), we obtain that (18′) has a countable set of periodic tori. Consequently, system (16) has a countable set of quasiperiodic motions.

In conclusion I express my gratitude to Academician L. S. Pontryagin for his attention to the work.

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* See also papers (7–9), devoted to the preservation of smooth invariant manifolds, in particular tori, under small smooth perturbations of vector fields.

Note: Figure translations are in progress. See original paper for figures.