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ON ONE POSSIBILITY OF GENERALIZING THE EQUATIONS OF THE GRAVITATIONAL FIELD

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The usual procedure for deriving the equations of the gravitational field from the principle of least action (see, for example, (^1)) consists in using a noninvariant density of the Lagrange function G, containing no second derivatives and differing from R by a total derivative. Equating to zero the first variation of the action

$$S = \frac{c^4}{8\pi k} \int G\sqrt{-g}\, d\Omega$$

gives Einstein' s equations. However, the sign of the second variation of the action then turns out to depend on the choice of the coordinate system, and one can only prove that it is possible to find such additional noninvariant conditions (for example, in (^1) $g_{0\alpha} = 0$, $|g_{\alpha\beta}| = \text{const}$), under which the second variation remains positive. One can obtain Einstein' s equations without introducing the function G by means of the variation

$$S = \frac{c^4}{8\pi k} \int R\sqrt{-g}\, d\Omega,$$

but in this case it is necessary to assume that on a hypersurface which, generally speaking, need not go to infinity, the variations δg_{ik} and $\partial \delta g_{ik}/\partial x^e$ simultaneously vanish. However, with such a method of variation the second variation of the action, although invariant, is likewise not sign-definite.

These circumstances led the author to reconsider the procedure of deriving the equations of the gravitational field from the principle of least action. The proposed method, while preserving the positivity and invariance of the second variation of the action, leads, however, to equations which differ from Einstein' s equations by the presence of an additional vector field A_i. In the present article the properties of the equations obtained and the possibilities of a physical interpretation of A_i as the vector potential of the electromagnetic field are discussed.

Let us compute the first variation of the action without restricting ourselves by any conditions on the variations of the field variables on the hypersurface. We shall denote the covariant derivative of a tensor by ∇_l; otherwise we shall use the notation adopted in (⌒1).

$$\frac{8\pi k}{c^4}\delta S = \delta \int R\sqrt{-g}\, d\Omega =$$
$$= \int d\Omega\sqrt{-g}\left[\left(R_{ik} - \frac{1}{2}g_{ik}R\right)\delta g^{ik} + \nabla_i \left(g^{li}g_{mn}\nabla_l \delta g^{mn} - \nabla_l \delta g^{li}\right)\right]. \quad (1)$$

The usual procedure consists in discarding the second term, as reducible to a surface integral. However, in order for the second term to vanish, it is by no means necessary to assume on the surface that $\nabla_i \delta g^{mn} = 0$. One may suppose that at every point of the volume there is a nonintegrable invariant relation between the variations, restricting the class of functions among which the minimum of the action is found,

$$g^{li}g_{mn}\nabla_i \delta g^{mn} - \nabla_i \delta g^{li} = 0. \quad (2)$$

Allowance for the additional relation (2) in the variation is made with the aid of Lagrange multipliers, after which all variations may be regarded as independent; on the surface $\delta g^{ik} = 0$, and we obtain the following equations for the gravitational field:

$$R_{ik} - \frac{1}{2}g_{ik}R - g_{ik}\nabla_l A^l + \frac{1}{2}\nabla_i A_k + \frac{1}{2}\nabla_k A_i = \frac{8\pi k}{c^4}T^*_{ik}. \quad (3)$$

The right-hand side of equation (3) is the tensor T_{ik}, written by analogy with Einstein' s equation, and denotes the contribution of matter. The addition of four functions A_k to the system of 10 field equations does not lead to underdetermination of the system, as may be verified by covariantly differentiating (3) with respect to x^k.

Calculation of the principal terms of the second variation, containing derivatives of δg^{ik} with respect to x^0, taking (2) into account, shows that in the proposed variational scheme the second variation is positive and invariant.

Assuming, for the case of a metric differing little from the Euclidean one,

$$g_{ik} = g^{(0)}_{ik} + h_{ik}$$

(the index 0, to simplify notation, will be omitted henceforth), and that together with h_{ik} the components of the vector A_i will be infinitesimals of the same order, we obtain the linear equations:

$$\frac{1}{2}g^{lm}\nabla_l\nabla_i h_{mk} + \frac{1}{2}g^{lm}\nabla_l\nabla_k h_{mi} - \frac{1}{2}g^{lm}\nabla_l\nabla_m h_{ik} - \frac{1}{2}g^{lm}\nabla_k\nabla_i h_{ml}$$
$$- \frac{1}{2}g_{ik}\left(g^{rs}g^{lm} - g^{ms}g^{lr}\right)\nabla_l\nabla_r h_{ms} - g_{ik}\nabla_l A^l + \frac{1}{2}\nabla_i A_k + \frac{1}{2}\nabla_k A_i = \frac{8\pi k}{c^4}T_{ik}. \quad (4)$$

By equations (4), h_{ik} are determined up to additive terms which may be obtained by means of an infinitesimal coordinate transformation $x'^i = x^i + \xi^i$

$$h'_{ik} = h_{ik} - \nabla_i\xi_k - \nabla_k\xi_i. \quad (5)$$

In addition, in the linear approximation there is the invariance

$$h'_{ik} = h_{ik} + g_{ik}\alpha, \qquad A'_i = A_i + \partial\alpha/\partial x^i. \quad (6)$$

The variables h_{ik} and A_i, satisfying (4), contain the arbitrariness of 4 functions in the choice of the coordinate system (5) and the arbitrariness of one function as a consequence of the invariance (6).

The equations for h_{ik} and A_i have the form of wave equations

$$\Box h_{ik} = -\frac{16\pi k}{c^4}T_{ik} + \frac{8\pi k}{c^4}g_{ik}T, \qquad \Box A_i = \frac{16\pi k}{c^4}\nabla_k T^k_i, \quad (7)$$

if the arbitrariness of the functions is eliminated by imposing 5 additional conditions, covariant in the linear approximation,

$$A_k + g^{mn}\nabla_n h_{mk} - {}^1/_2 g^{mn}\nabla_k h_{mn} = 0, \qquad \nabla_l A^l = 0. \quad (8)$$

It is obvious that, in the linear approximation, T^{ik} always satisfies the equation

$$\nabla_i\nabla_k T^{ik} = 0. \quad (9)$$

There remains arbitrariness for the transformations (5) and (6) with ξ_i and α satisfying the equations:

$$\Box\xi_i = 0, \qquad \Box\alpha = 0, \quad (10)$$

which may be used to eliminate 5 of the remaining 9 independent components of the gravitational wave.

* The addition of a vector field analogous to (3) was considered by Infeld and Plebanski (2) as a method for jointly solving the equations of gravitation and the equations of motion.

The remaining 4 components cannot be eliminated by any transformations. Two components of the wave coincide with gravitational waves when $A_i = 0$. The other 2 components correspond to transverse waves of the vector A_i. In this case the individual components A_i are different from zero and are found from conditions (8).

A natural question is whether the vector A_i can be interpreted as the vector potential of the electromagnetic field.

It is shown in the work that, in the absence of matter, A_i satisfies the wave equation. In the linear approximation the gradient invariance (6) holds. Waves of the vector A_i are transverse (8).

For a model of matter corresponding to the classical motion of separate uncharged material particles, the Lagrangian function is known, and on the right-hand side of equation (3) there must stand the energy-momentum tensor of matter, whose covariant derivative is equal to zero. It follows from (7) that in this case A_i vanishes.

Taking the point of view that A_i is the electromagnetic potential arising in the process of variation, we cannot at the same time include A_i, as is usually done, in the Lagrangian function and assume that for charged matter the right-hand side of (3) is the energy-momentum tensor of matter and of the electromagnetic field. Moreover, identifying equations (7) of the linear approximation with the equations of the electromagnetic field, we are compelled to require that the derivative of the right-hand side of the field equations T^{ik} be proportional to the four-dimensional current density of charged particles j^k

$$\nabla_i T^{ik} = a j^k, \tag{11}$$

where a is a certain unknown constant coefficient determining the proportionality between the vector potential of the electromagnetic field and the vector A_i, which has dimension 1/cm. Equation (9), taking (11) into account, expresses the law of conservation of charge.

The author has not succeeded in constructing an explicit expression T^{ik} for charged matter satisfying (11). Therefore the question of the possibility of interpreting A_i as the vector potential of the electromagnetic field remains open.

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CITED LITERATURE

[1] L. D. Landau, E. M. Lifshitz, *The Classical Theory of Fields*, Moscow, 1960.
[2] L. Infeld, E. Plebanski, *Motion and Relativity*, IL, 1962.

Note: Figure translations are in progress. See original paper for figures.