LACUNARY SYSTEMS

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 513.88 : 513.83

MATHEMATICS

B. E. VEITS

LACUNARY SYSTEMS

(Presented by Academician V. I. Smirnov on 26 V 1967)

In this note the concept of lacunarity is generalized to systems of elements of Banach spaces, and for these systems some generalizations and analogues are established of properties known for lacunary systems of functions.

1. Let a separable Banach space E over the field of complex numbers be densely embedded in a Hilbert space H, $E \neq H$, and $\|x\|_H \leq \|x\|_E$. Each element $y \in H$ generates a functional $f_y \in E^*$, acting by the formula $f_y(x) = (x, y)$. Moreover $\|f_y\|_{E^*} \leq \|y\|_H$. Denote by E_1 the closure of H in the norm of the space E^*. After the corresponding identifications one may assume that

$$E \subset H \subset E_1 \subset E^*.$$

Let (x_k) be a Riesz basis sequence in H, i.e. it is a Riesz basis ([1]) in the closure (in the norm of H) of its linear span $[x_k]_H$. Generalizing the known definition ([1]), we shall call this system E-lacunary if $(x_k) \subset E$ and, for any sequence $(a_k) \in l^2$, the series

$$\sum_{k=1}^{\infty}{}_H a_k x_k$$

converges in the norm of the space H to some element of the space E (the subscript H or E at the summation sign indicates in which norm the series converges).

From the definition of lacunarity it follows that

Corollary 1. *An E-lacunary system (x_k) is a Bessel system ([2]) in its linearly closed span $[x_k]_E$, i.e. for every $x \in [x_k]_E$ the series*

$$\sum_{k=1}^{\infty} |f_k(x)|^2$$

converges, where (f_k) is the biorthogonal system of linear functionals from $[x_k]_E^*$.

Let us note that if $(x_i, y_k) = \delta_{ik}$, $y_k \in [x_k]_H$; $i, k = 1, 2, ...$, then

$$f_k = f_{y_k}: \quad f_k(x) = (x, y_k), \qquad k = 1, 2, ...$$

Theorem 1. *In order that an H-basis Riesz system $(x_k) \subset E$ be E-lacunary, it is necessary and sufficient that there exist a number $l > 0$, independent of the numbers $\xi_1, \xi_2, \dots, \xi_n$ and of the natural number n, such that*

$$\left\| \sum_{k=1}^{n} \xi_k x_k \right\|_E \leq l \left(\sum_{k=1}^{n} |\xi_k|^2 \right)^{1/2}. \tag{1}$$

Theorem 2. *In order that the system (x_k) be E-lacunary, it is necessary and sufficient that, for every $f \in E^*$, the series*

$$\sum_{k=1}^{\infty} |f(x_k)|^2 \tag{2}$$

converge.

Proof. Necessity. According to Theorem 1, there exists a number $l > 0$ such that, for every $f \in E^*$,

$$\left\| \sum_{k=1}^{n} \overline{f(x_k)}\, x_k \right\|_E \leq l \left(\sum_{k=1}^{n} |f(x_k)|^2 \right)^{1/2}.$$

On the other hand,

$$\sum_{k=1}^{n} |f(x_k)|^2 = f\left(\sum_{k=1}^{n} \overline{f(x_k)} x_k \right) \leq \|f\| \left\| \sum_{k=1}^{n} \overline{f(x_k)} x_k \right\| \leq l \|f\| \left(\sum_{k=1}^{n} |f(x_k)|^2 \right)^{1/2}.$$

Therefore

$$\left(\sum_{k=1}^{n} |f(x_k)|^2 \right)^{1/2} \leq l \|f\|,$$

and, consequently, the series (2) converges.

Sufficiency. From the convergence of the series (2) for every $f \in E^*$ it follows, as is known [(1)] (p. 435), that there exists a number $l > 0$ such that

$$\sum_{k=1}^{\infty} |f(x_k)|^2 \leq l^2 \|f\|^2.$$

Let $\xi_1, \xi_2, \dots, \xi_n$ be arbitrary complex numbers. There exists a functional $f_0 \in E^*$, $\|f_0\| = 1$, such that

$$\left\| \sum_{k=1}^{n} \xi_k x_k \right\|_E = \sum_{k=1}^{n} \xi_k f_0(x_k) \leq$$

$$\leq \left(\sum_{k=1}^{n} |f_0(x_k)|^2 \right)^{1/2} \left(\sum_{k=1}^{n} |\xi_k|^2 \right)^{1/2} \leq l \|f_0\| \left(\sum_{k=1}^{n} |\xi_k|^2 \right)^{1/2}.$$

The latter, according to Theorem 1, means the E-lacunarity of (x_k).

Corollary 2. An E-lacunary system (x_k) is a Hilbert system [(2)] in $[x_k]_E$, i.e. for any sequence $(a_k) \in l^2$ there exists a unique element $x \in [x_k]_E$ such that $f_k(x) = a_k$, $k = 1, 2, \dots$.

Corollary 3. An E-lacunary system (x_k) forms a Riesz basis in the subspace $[x_k]_E$, which is isomorphic to l^2 [(2)].

Corollary 4. If the system (x_k) is E-lacunary, then the system (f_k) forms a Riesz basis in the subspace $[f_k]_{E^*}$.

Theorem 3. In the space $C[0,1]$ of functions continuous on $[0,1]$ there do not exist C-lacunary systems.

Proof. If (x_k) is a Riesz basic sequence in L^2, then [(1)] there exists a number $m > 0$ such that

$$m \leq \|x_k\|_{L^2} = \left(\int_0^1 |x_k(t)|^2 \, dt \right)^{1/2}, \qquad k = 1, 2, \dots \tag{3}$$

If this system were C-lacunary, then for every $f \in C^*$

$$\sum_{k=1}^{n} |f(x_k)|^2 \leq l^2 \|f\|^2.$$

In particular, for every $t \in [0,1]$ we have

$$\sum_{k=1}^{n} |x_k(t)|^2 \leq l^2$$

uniformly in n and t. Consequently, the series

$$\sum_{k=1}^{\infty} \int_0^1 |x_k(t)|^2 \, dt$$

would converge, which would contradict the inequalities (3).

Theorem 4. If the system (x_k) is E-lacunary, then

$$\left(\sum_{k=1}^{n} |\xi_k|^2 \right)^{1/2} \leq l \left\| \sum_{k=1}^{n} \xi_k f_k \right\|_{E^*}, \tag{4}$$

where l depends neither on n nor on the choice of the numbers $\xi_1, \xi_2, \dots, \xi_n$.

Theorem 5. If (x_k) is an E-lacunary system, then the subspace $[x_k]_E$ has a complement in the space E.

Theorem 6. If (x_k) is an E-lacunary system, then the subspace $[f_k]_{E^*}$ has a complement in the space E^*.

2. Let the Banach space E be densely embedded in a Banach space B, which in turn is densely embedded in a Hilbert space H, and for all $x \in E$

$$\|x\|_H \leq \|x\|_B \leq \|x\|_E.$$

If by B_1 we denote the closure of H in the norm in B^*, then one may assume that

$$E \subset B \subset H \subset B_1 \subset E_1 \subset E^*.$$

Since H is reflexive, it follows from the results of S. G. Krein and Yu. I. Petunin ([3]) (p. 106) that the spaces H and E_1 are related, i.e., there exists a continuous normal scale $\{E_\alpha\}$ of Banach spaces E_α, $0 \leq \alpha \leq 1$, $E_0 = H$, connecting the space E_1 with E_0 and satisfying the conditions:

1°. $E_\alpha \subset E_\beta$ for $\alpha < \beta$.

2°. $\|x\|_\beta \leq \|x\|_\alpha$ for $\alpha < \beta$ and $x \in E_\alpha$.

3°. $\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu$, where $\alpha \leq \beta \leq \gamma$ and $\mu + \nu = 1$.

In the following theorem the well-known inequality of Khinchin ([1]) (p. 285) is generalized.

Theorem 7. *If the system $(x_k) \subset E$ is B-lacunary and among the scales connecting E_1 with E_0 there exists a continuous normal scale $\{E_\alpha\}$ such that $E_{\alpha_0} \subset B_1$ at least for one $\alpha_0 \in (0,1)$, then*

$$\left(\sum_{k=1}^{n} |\xi_k|^2\right)^{1/2} \leq C \left\|\sum_{k=1}^{n} \xi_k f_k\right\|_{E^*}.$$

Proof. Since the system (y_k) in $[x_k]_H$ is a Riesz basis, we have

$$\left\|\sum_{k=1}^{n} \xi_k y_k\right\|_H \leq m \left(\sum_{k=1}^{n} |\xi_k|^2\right)^{1/2}. \tag{5}$$

Next, we have

$$\sum_{k=1}^{n} |\xi_k|^2 = \left(\sum_{k=1}^{n} \xi_k x_k, \sum_{k=1}^{n} \xi_k y_k\right) \leq \left\|\sum_{k=1}^{n} \xi_k x_k\right\|_B \left\|\sum_{k=1}^{n} \xi_k f_k\right\|_{B^*}, \tag{6}$$

$$\left\|\sum_{k=1}^{n} \xi_k f_k\right\|_{B^*} \leq \left\|\sum_{k=1}^{n} \xi_k f_k\right\|_{\alpha_0} \leq \left\|\sum_{k=1}^{n} \xi_k f_k\right\|_H^{\mu} \left\|\sum_{k=1}^{n} \xi_k f_k\right\|_{E_1}^{\nu} =$$

$$= \left\|\sum_{k=1}^{n} \xi_k y_k\right\|_H^{\mu} \left\|\sum_{k=1}^{n} \xi_k f_k\right\|_{E^*}^{\nu} \leq m^{\mu} \left(\sum_{k=1}^{n} |\xi_k|^2\right)^{\mu/2} \left\|\sum_{k=1}^{n} \xi_k f_k\right\|_{E^*}^{\nu}. \tag{7}$$

Taking into account inequalities (1), (6), and (7), we obtain:

$$\sum_{k=1}^{n} |\xi_k|^2 \leq l \left(\sum_{k=1}^{n} |\xi_k|^2\right)^{1/2} m^{\mu} \left(\sum_{k=1}^{n} |\xi_k|^2\right)^{\mu/2} \left\|\sum_{k=1}^{n} \xi_k f_k\right\|_{E^*}^{\nu}.$$

Putting $l^{1/\nu} m^{\mu/\nu} = C$, we obtain, taking into account that $\mu + \nu = 1$,

$$\left(\sum_{k=1}^{n} |\xi_k|^2\right)^{1/2} \leq C \left\|\sum_{k=1}^{n} \xi_k f_k\right\|_{E^*}.$$

Corollary 5. *Under the conditions of the preceding theorem the system (f_k) forms a Riesz basis in the subspace $[f_k]_{E^*}$.*

The following theorem generalizes the well-known theorem of Banach ([6]).

Theorem 8. *If $(x_k) \subset E$ is a basic Riesz sequence in $[x_k]_H$, then the following three statements are equivalent:*

1°. *For every $f \in [f_k]_{E^*}$ the series $\sum_{k=1}^{\infty} |f(x_k)|^2$ converges.*

2°. *There exists a number l, independent of n and of the numbers $\xi_1, \xi_2, \dots, \xi_n$, such that*

$$\left(\sum_{k=1}^{n} |\xi_k|^2\right)^{1/2} \leq l \left\| \sum_{k=1}^{n} \xi_k f_k \right\|_{E^*}. \tag{8}$$

3°. *For any sequence $(a_k) \in l^2$ there exists an element $x \in E$ such that $f_k(x) = a_k,\ k = 1, 2, \dots$.*

Proof. If 1° is fulfilled, then

$$\sum_{k=1}^{\infty} |f(x_k)|^2 \leq l^2 \|f\|^2 \quad \text{for } f \in [f_k]_{E^*}.$$

If in this inequality we put $f = \sum_{k=1}^{n} \xi_k f_k$, then we obtain assertion 2°. If, on the other hand, we assume the validity of assertion 2°, then for $f \in [f_k]_{E^*}$ there is a sequence $(f^{(n)})$ of elements of $[f_k]_{E^*}$ such that

$$f^{(n)} = \sum_{k=1}^{p_n} \alpha_k^{(n)} f_k \xrightarrow[n\to\infty]{E^*} f, \qquad \|f^{(n)}\|_{E^*} \leq C.$$

Taking 2° into account, we obtain

$$\left(\sum_{k=1}^{p_n} |\alpha_k^{(n)}|^2\right)^{1/2} \leq l \|f^{(n)}\|_{E^*} \leq lC,$$

and since there exists $\lim_{n\to\infty} \alpha_k^{(n)} = \lim_{n\to\infty} f^{(n)}(x_k) = f(x_k)$, it is then easy to obtain the convergence of the series in l^0. Thus the equivalence of assertions 1° and 2° is established. The method of proof of the equivalence 2° ⇔ 3° is the same as in [6].

Corollary 6. *Under the conditions of Theorem 7, for every sequence $(a_k) \in l^2$ there exists an element $x \in E$ such that*

$$f_k(x) = a_k, \qquad k = 1, 2, \dots$$

Corollary 7. *If an orthonormal system $(e_k) \subset C[0,1]$ is L^p-lacunary for some $p > 2$, then the Lebesgue functions of this system cannot be uniformly bounded.*

Murmansk State
Pedagogical Institute

Received
11 V 1967

REFERENCES

1. S. Kaczmarz, H. Steinhaus, *Theory of Orthogonal Series*, Moscow, 1958.
2. B. E. Veits, Izv. Vyssh. Uchebn. Zaved., Matem., No. 2, 7 (1965).
3. S. G. Krein, Yu. I. Petunin, UMN, 21, issue 2 (1966).
4. N. K. Bari, *Trigonometric Series*, Moscow, 1961.
5. B. E. Veits, Izv. Vyssh. Uchebn. Zaved., Matem., No. 4, 24 (1965).
6. S. Banach, Studia Math., 2, 207 (1930).

Note: Figure translations are in progress. See original paper for figures.