ON THE ORDER OF APPROXIMATION OF A FUNCTION BY ZYGMUND NORMAL MEANS

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Abstract

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Let L_p $(1 \leqslant p \leqslant \infty)$ denote the space of measurable 2π-periodic functions for which

$$\|f(x)\|_{L_p} = \left\{ \int_0^{2\pi} |f(x)|^p \, dx \right\}^{1/p} < \infty \quad (1 \leqslant p < \infty),$$

and, for $p = \infty$,

$$\operatorname*{vrai\,sup}_x |f(x)| < \infty.$$

For each function $f(x) \in L_p$ with Fourier series

$$\sum_{\nu=0}^{\infty} A_\nu(x), \quad \left(A_0(x) \equiv \frac{a_0}{2}, \quad A_\nu(x) = a_\nu \cos \nu x + b_\nu \sin \nu x, \quad \nu = 1, 2, \ldots \right),$$

for any natural k we consider the operator

$$Z_n^{(k)}(f; x) = \sum_{\nu=0}^{n} \left(1 - \frac{\nu^k}{(n+1)^k} \right) A_\nu(x),$$

which is usually called the Zygmund normal mean of order k.

Numerous works have been devoted to the investigation of the order of approximation of each individual function $f(x) \in L_p$ by the operators $Z_n^{(k)}(f; x)$, depending on its structural properties. At the same time, as Zygmund already showed in (^{1}), there is a substantial difference in considering this question for the cases of even and odd k.

If k is an even number, and the function $f(x)$ has a $(k-1)$-st absolutely continuous derivative and $\|f^{(k)}(x)\|_{L_\infty} < \infty$, then (see (^{1}))

$$\|f(x) - Z_n^{(k)}(f; x)\|_{L_\infty} \leqslant c_k (n+1)^{-k} \|f^{(k)}(x)\|_{L_\infty}. \qquad (1)$$

Another proof of this Zygmund inequality was given by the author in (^{8}). It is based on the obvious estimate

$$\|f(x)-Z_n^{(k)}(f;x)\|_{L_p} \leqslant (1+\|Z_n^{(k)}\|)\|f(x)-T_n(x)\|_{L_p}+(n+1)^{-k}\left\|\frac{d^k}{dx^k}T_n(x)\right\|_{L_p} \tag{2}$$

valid for any trigonometric polynomial $T_n(x)$ (see (^{8}) for $p=\infty$ and (^{7}), p. 591, for any $1\leqslant p\leqslant\infty$).

From inequality (1) and one result of S. B. Stechkin (^{4}) it follows that, for any continuous function $f(x)$,

$$\|f(x)-Z_n^{(k)}(f;x)\|_{L_\infty} \leqslant c_k\omega_k\left(f;\frac{1}{n+1}\right)_{L_\infty}, \tag{3}$$

where

$$\omega_k(f;h)_{L_p}=\sup_{|t|\leq h}\|\Delta_t^k f(x)\|_{L_p}=\sup_{|t|\leq h}\left\|\sum_{\nu=0}^{k}(-1)^{k-\nu}\binom{k}{\nu}f(x+\nu t)\right\|_{L_p}.$$

Estimate (3) and the analogous estimate for the case where $f(x)\in L_p$ $(1\leqslant p\leqslant\infty)$,

$$\|f(x)-Z_n^{(k)}(f;x)\|_{L_p} \leqslant c_k\omega_k\left(f;\frac{1}{n+1}\right)_{L_p}, \tag{4}$$

follow from (2), if one uses the following inequality for pro-

derivatives of a trigonometric polynomial (see (^{5,6})):

$$\left\|\frac{d^k}{dx^k}T_n(x)\right\|_{L_p} \ll \left(\frac{n}{2}\right)^k\left\|\Delta_{\pi/2n}^k T_n(x)\right\|_{L_p}. \tag{5}$$

R. M. Trigub (^{10}) and V. V. Zhuk (^{11}) independently showed that, for even k, for any function $f(x)\in L_p$ the converse to inequality (4) is also true, i.e.,

$$\omega_k\left(f;\frac{1}{n+1}\right)_{L_p} \ll c_k\left\|f(x)-Z_n^{(k)}(f;y)\right\|_{L_p} \qquad (1\leq p\leq\infty). \tag{6}$$

Thus, for any function $f(x)\in L_p$ $(1\leq p\leq\infty)$, for even k the order relation * holds:

$$\left\|f(x)-Z_n^{(k)}(f;x)\right\|_{L_p} \asymp \omega_k\left(f;\frac{1}{n+1}\right)_{L_p}. \tag{7}$$

In the case when k is an odd number and $1<p<\infty$, relation (7) also remains valid. Estimate (4) in this case for $k=1$ was obtained by P. L. Ul' yanov (^{9}), and for other values of k by Yu. A. Ponomarenko (12). Inequality (6) can be obtained with the aid of the identity

$$\frac{d^k}{dx^k} Z_n^{(k)}(f;x) = (-1)^{k/2}(n+1)^k \left\{ \widetilde{Z}_n^{(k)}(f;x) - \widetilde{Z}_{n,1}^{(k)}(f;x) \right\}, \tag{8}$$

where $\widetilde{Z}_n^{(k)}(f;x)$ are the Zygmund normal means for the series conjugate to the Fourier series of the function $f(x)$, and $\widetilde{Z}_{n,1}^{(k)}(f;x) = Z_n^{(k)}(\widetilde{Z}_n^{(k)}(f);x)$. Indeed, for any h $\left(0 < h \leq \frac{1}{n+1}\right)$,

$$\left\|\Delta_h^k f(x)\right\|_{L_p} \leq \left\|\Delta_h^k f(x) - \Delta_h^k Z_n^{(k)}(f;x)\right\|_{L_p} + h^k \left\|\frac{d^k}{dx^k} Z_n^{(k)}(f;x)\right\|_{L_p} \leq$$

$$\leq 2^k \left\|f(x) - Z_n^{(k)}(f;x)\right\|_{L_p} + \left\|\widetilde{Z}_n^{(k)}(f;x) - \widetilde{Z}_{n,1}^{(k)}(f;x)\right\|_{L_p}.$$

Since $1 < p < \infty$, the function $\tilde{f}(x)$, conjugate to $f(x)$, also belongs to the space L_p, and, by the well-known inequality of M. Riesz (see (13), p. 149), we find

$$\left\|\Delta_h^k f(x)\right\|_{L_p} \leq 2^k \left\|f(x) - Z_n^{(k)}(f;x)\right\|_{L_p} + \left\|\tilde{f}(x) - \widetilde{Z}_n^{(k)}(f;x)\right\|_{L_p} +$$

$$+ \left\|\tilde{f}(x) - \widetilde{Z}_{n,1}^{(k)}(f;x)\right\|_{L_p} \leq c_{p,k} \left\{ \left\|f(x) - Z_n^{(k)}(f;x)\right\|_{L_p} + \left\|f(x) - Z_{n,1}^{(k)}(f;x)\right\|_{L_p} \right\},$$

where

$$Z_{n,1}^{(k)}(f;x) = Z_n^{(k)}(Z_n^{(k)}(f);x) = \sum_{\nu=0}^{n} \left(1 - \frac{\nu^k}{(n+1)^k}\right)^2 A_\nu(x).$$

From this it is easy to obtain

$$\left\|\Delta_h^k f(x)\right\|_{L_p} \ll M_{p,k} \left\|f(x) - Z_n^{(k)}(f;x)\right\|_{L_p}.$$

From this estimate we obtain (6), with a constant depending not only on k, but also on p.

For $p = \infty$ or $p = 1$ and odd k, the order relation (7), as is known, does not hold. In these cases, when studying the order of approximation of a function $f(x)$ by the operators $Z_n^{(k)}(f;x)$, as has been shown in a number of works (see, for example, (1–4)), it is important to take into account the structural properties not only of the function $f(x)$ itself, but also the properties of the function $\tilde{f}(x)$ conjugate to it.

Below the following assertion is established concerning the approximation of a function $f(x) \in L_p$ $(1 \leq p \leq \infty)$ by the Zygmund normal means $Z_n^{(k)}(f;x)$ for the case when k is any odd number.

* The relation $u \asymp v$ means that $c_1 v \leq u \leq c_2 v$, where $c_1 > 0$, $c_2 > 0$ are some constants.

Theorem. If $f(x) \in L_p$ $(1 \leq p \leq \infty)$ has Fourier series $\sum_{\nu=0}^{\infty} A_\nu(x)$, then for any odd k the following order relation holds:

$$\left\| f(x) - Z_n^{(k)}(f;x) \right\|_{L_p} \preccurlyeq \omega_{k+1}\left(f; \frac{1}{n+1}\right)_{L_p} + \omega_{k+1}\left(\widetilde{F}; \frac{1}{n+1}\right)_{L_p}(n+1), \quad (9)$$

where $\widetilde{F}(x)$ is a function having Fourier series $\sum_{\nu=1}^{\infty} \frac{1}{\nu} A_\nu(x)$.*

The proof of the theorem is based on the following, easily verified identities:**

$$(n+1)^{-k-1} \frac{d^{k+1}}{dx^{k+1}} Z_n^{(k)}(f;x) = (-1)^{(k-1)/2} \left[Z_n^{(k)}(f;x) - Z_{n,1}^{(k)}(f;x) - Z_n^{(k)}(Z_n^{(1)}(f); x) + Z_{n,1}^{(k)}(Z_n^{(1)}(f);x) \right], \quad (10)$$

$$(n+1)^{-k} \frac{d^{k+1}}{dx^{k+1}} Z_n^{(k+1)}(\widetilde{F};x) = (-1)^{(k-1)/2} \left[Z_n^{(k+1)}(f;x) - Z_n^{(k+1)}(Z_n^{(k)}(f);x) \right], \quad (11)$$

$$Z_n^{(k+1)}(f;x) = Z_n^{(k)}(f;x) + Z_n^{(1)}(f;x) - Z_n^{(1)}(Z_n^{(k)}(f);x). \quad (12)$$

Let us first establish that for any p $(1 \leq p \leq \infty)$ and odd k

$$\left\| f(x) - Z_n^{(k)}(f;x) \right\|_{L_p} \preccurlyeq c_k \left\{ \omega_{k+1}\left(f; \frac{1}{n+1}\right)_{L_p} + (n+1)\omega_{k+1}\left(\widetilde{F}; \frac{1}{n+1}\right)_{L_p} \right\}. \quad (13)$$

Obviously,

$$\left\|f(x)-Z_n^{(k)}(f;x)\right\|_{L_p} \le \left\|f(x)-Z_n^{(k+1)}(f;x)\right\|_{L_p} + \left\|Z_n^{(k)}(f-Z_n^{(k+1)}(f);x)\right\|_{L_p}$$
$$+\left\|Z_n^{(k+1)}(f;x)-Z_n^{(k)}(Z_n^{(k+1)}(f);x)\right\|_{L_p} = U_1+U_2+U_3. \tag{14}$$

Since $k+1$ is an even number, by (4) and the boundedness of the norm of the operator $Z_n^{(k)}(f;x)$, we find

$$U_1 \preccurlyeq c_k\omega_{k+1}\left(f;\frac{1}{n+1}\right)_{L_p}, \qquad U_2 \preccurlyeq b_k\omega_{k+1}\left(f;\frac{1}{n+1}\right)_{L_p}. \tag{15}$$

Using identity (11) and inequalities (5) and (4), we obtain

$$\begin{aligned} U_3 &\preccurlyeq (n+1)^{-k}\left\|\frac{d^{k+1}}{dx^{k+1}}Z_n^{(k+1)}(\widetilde{F};x)\right\|_{L_p} \\ &\preccurlyeq B_k(n+1)\omega_{k+1}\left(Z_n^{(k+1)}(\widetilde{F});\frac{1}{n+1}\right)_{L_p} \\ &\preccurlyeq M_k(n+1)\left\{\omega_{k+1}\left(\widetilde{F};\frac{1}{n+1}\right)_{L_p}+\left\|\widetilde{F}(x)-Z_n^{(k+1)}(\widetilde{F};x)\right\|_{L_p}\right\} \\ &\preccurlyeq M_k'(n+1)\omega_{k+1}\left(\widetilde{F};\frac{1}{n+1}\right)_{L_p}. \end{aligned} \tag{16}$$

From estimates (15), (16), and (14), inequality (13) follows.

We shall now show that for any odd k the inequalities

$$\omega_{k+1}\left(f;\frac{1}{n+1}\right)_{L_p} \preccurlyeq c_k\left\|f(x)-Z_n^{(k)}(f;x)\right\|_{L_p}, \tag{17}$$

$$\omega_{k+1}\left(\widetilde{F};\frac{1}{n+1}\right)_{L_p} \preccurlyeq \frac{c_k}{n+1}\left\|f(x)-Z_n^{(k)}(f;x)\right\|_{L_p}. \tag{18}$$

* We note that in the special case when $k=1$ (Fejér sum), relation (9) was announced, without indication of its proof, by V. V. Zhuk at the interuniversity summer scientific school on summability theory, Sverdlovsk, July 1967.

** Such identities, with an indication of their application to estimates of deviations of functions from their normal Zygmund means, were earlier presented by the author in a report at the interuniversity seminar on function theory, Dnepropetrovsk, April 21, 1967, and at the summer scientific school, Sverdlovsk, July 1967.

Let $0 < h \leq \dfrac{1}{n+1}$. Using identity (10), we find

$$\begin{aligned}\|\Delta_h^{k+1} f(x)\|_{L_p} &\leq 2^{k+1}\|f(x) - Z_n^k(f;x)\|_{L_p} + \|\Delta_h^{k+1} Z_n^{(k)}(f;x)\|_{L_p} \\ &\leq 2^{k+1}\|f(x) - Z_n^{(k)}(f;x)\|_{L_p} + h^{k+1}\left\|\frac{d^{k+1}}{dx^{k+1}} Z_n^{(k)}(f;x)\right\|_{L_p} \\ &\leq 2^{k+1}\|f(x) - Z_n^{(k)}(f;x)\|_{L_p} \\ &\quad + \|Z_n^{(k)}(f;x) - Z_{n,1}^{(k)}(f;x) - Z_n^k(Z_n^{(1)}(f);x) + Z_{n,1}^{(k)}(Z_n^{(1)}(f);x)\|_{L_p}.\end{aligned} \tag{19}$$

Adding and subtracting under the norm sign the difference $f(x) - Z_n^{(1)}(f;x)$, we obtain

$$\begin{aligned}&\|Z_n^{(k)}(f;x) - Z_{n,1}^{(k)}(f;x) - Z_n^{(k)}(Z_n^{(1)}(f);x) + Z_{n,1}^{(k)}(Z_n^{(1)}(f);x)\|_{L_p} \\ &\leq \|f(x) - Z_n^{(k)}(f;x)\|_{L_p} + \|f(x) - Z_{n,1}^{(k)}(f;x)\|_{L_p} \\ &\quad + \|Z_n^{(k)}(f - Z_n^{(k)}(f);x)\|_{L_p} + \|Z_n^{(1)}(f - Z_n^{(k)}(f);x)\|_{L_p}.\end{aligned} \tag{20}$$

Since, obviously,

$$\|f(x) - Z_{n,1}^{(k)}(f;x)\|_{L_p} \leq B_k\|f(x) - Z_n^{(k)}(f;x)\|_{L_p},$$

it follows from (19) and (20) that we arrive at the estimate

$$\|\Delta_h^{k+1} f(x)\|_{L_p} \leq c_k\|f(x) - Z_n^{(k)}(f;x)\|_{L_p}.$$

In view of the fact that this inequality is valid for every $0 < h \leq \dfrac{1}{n+1}$, we obtain estimate (17). Similarly, using identity (11) and identity (12) as applied to the function $\widetilde{F}(x)$, for every $h\left(0 < h \leq \dfrac{1}{n+1}\right)$,

$$\begin{aligned}\|\Delta_h^{k+1}\widetilde{F}(x)\|_{L_p} &\leq 2^{k+1}\|\widetilde{F}(x) - Z_n^{(k+1)}(\widetilde{F};x)\|_{L_p} + h^{k+1}\left\|\frac{d^{k+1}}{dx^{k+1}} Z_n^{(k+1)}(\widetilde{F};x)\right\|_{L_p} \\ &\leq 2^{k+1}\|\widetilde{F}(x) - Z_n^{(k)}(\widetilde{F};x) + Z_n^{(1)}(\widetilde{F};x) - Z_n^{(1)}(Z_n^{(k)}(\widetilde{F});x)\|_{L_p} \\ &\quad + \frac{1}{n+1}\|Z_n^{(k+1)}(f;x) - Z_n^{(k+1)}(Z_n^{(k)}(f);x)\|_{L_p} \\ &\leq 2^{k+1}\|\varphi_k(x) - Z_n^{(1)}(\varphi_k;x)\|_{L_p} + \frac{1}{n+1}\|Z_n^{(k+1)}(f - Z_n^{(k)}(f);x)\|_{L_p},\end{aligned}$$

where $\varphi_k(x) = \widetilde{F}(x) - Z_n^{(k)}(\widetilde{F}; x)$. Applying to the function $\varphi_k(x)$ the known theorem of Alexits (3), we obtain that

$$\|\varphi_k(x) - Z_n^{(1)}(\varphi_k; x)\|_{L_p} \leq C\frac{\|\widetilde{\varphi}_k(x)\|_{L_p}}{n+1} = C\frac{\|f(x) - Z_n^{(k)}(f; x)\|_{L_p}}{n+1},$$

where $\widetilde{\varphi}_k(x)$ is the function conjugate to $\varphi_k(x)$. Thus,

$$\|\Delta_h^{k+1}\widetilde{F}(x)\|_{L_p} \leq \frac{c_k}{n+1}\|f(x) - Z_n^{(k)}(f; x)\|_{L_p}.$$

From this inequality (18) follows. From the estimates (17), (18), and (13) we arrive at the order relation (9).

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Note: Figure translations are in progress. See original paper for figures.

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