LOCALLY HOMOTOPICALLY UNKNOTTED EMBEDDINGS OF MANIFOLDS

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LOCALLY HOMOTOPICALLY UNKNOTTED EMBEDDINGS OF MANIFOLDS

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In this note we give a strengthening of a theorem proved by Bryant and Seebeck, which belongs to the type of the so-called engulfing lemmas (^{1}). Bryant and Seebeck consider embeddings of an ANR in R^n in codimensions greater than two. We consider embeddings of compact manifolds, but in all codimensions, and obtain one additional strengthening. In the second section we consider certain properties of embeddings which express, in a certain sense, "local homotopic unknottedness." They are used to formulate the main result and its applications. The proofs use only standard homotopy technique and Stallings' technique (^{2}), and we confine ourselves to brief indications.

We denote by B^n the unit ball of Euclidean space R^n and assume that B^{n-1} is embedded in B^n in the standard way. By $[X]$ and $\operatorname{Int} X$ are denoted the closure and the interior of a set X in R^n; by M and $\dot{M}$, the boundary and the interior of a manifold M; by Λ, the empty set; by $O_\varepsilon(X)$, the ε-neighborhood of the set X in R^n. If $x \in M$, where $M \subset R^n$ is a manifold, then by ε_x is denoted such a number that in M there is a neighborhood H_x, homeomorphic to a closed ball, such that

$$H_x \cap [O_\varepsilon(x)] = M \cap [O_\varepsilon(x)].$$

If H, U, V are neighborhoods of a manifold M in R^n, then a homotopy $\varphi_t : U \to H$ is called a **strong deformation** of U into H in V, if there exists a neighborhood W such that: 1) $\varphi_0 = 1$; 2) $\varphi_t|_W = 1$; 3) $\varphi_1 U \subset V$; 4) $\varphi_t(U \setminus W) \cap W = \Lambda$, $0 \leq t \leq 1$.

If $q : \widetilde{M} \to R^n$, $q\widetilde{M} = M$, is an embedding of a k-dimensional manifold, then q is called **locally abelian** if for every point $x \in M$ and for every ε, $0 < \varepsilon < \varepsilon_x$, there exists a $\delta > 0$ such that every loop in $O_\delta(x) \setminus M$ is null-homotopic in $O_\varepsilon(x) \setminus M$ if and only if it is homologous to zero in $O_\varepsilon(x) \setminus M$. The embedding is called **homotopically locally unknotted** if it is: 1) locally abelian and 2) in the case $k = n-2$, every i-dimensional spheroid in $O_\varepsilon(x) \setminus M$ is null-homotopic in $O_\varepsilon(x) \setminus M$ for $2 \leq i \leq n$. In the case $k \neq n-2$, a locally abelian embedding is called **locally 1-connected**.

All these properties have been considered many times in the literature. In particular, the condition of local abelianness was introduced by Harrold in (^{3}).

1. Strengthening of the Bryant-Seebeck theorem and the theorem on the union of cells. Let $q : \widetilde{M} \to R^n$ be an embedding of a k-dimensional manifold, $M = q\widetilde{M}$, and let $q_\alpha : \widetilde{M} \to R^n$ be a certain collection of embeddings, $M_\alpha = q_\alpha \widetilde{M}$.

Definition. The collection $Q = \{q_\alpha\}$ will be called a **class of embeddings close to** q if for every $\varepsilon > 0$ there exist $\delta > 0$ and $\eta > 0$ such that for every embedding $q_\alpha \in Q$ which is δ-close to q, there is a strong ε-deformation of $O_\eta(M)$ into $O_\varepsilon(M)$ in any neighborhood of M_α.

Lemma 1. *Let the embedding q be locally homotopically unknotted and let $Q = \{q_\alpha\}$ be some class of embeddings close to q which are locally homotopically unknot-*

embeddings. Then for every $\varepsilon > 0$ there exist $\eta > 0$ and $\delta > 0$ such that for every embedding $q_\alpha \in Q$ that is δ-close to q there exists an ε-deformation

$$\varphi_t : R^n \setminus M_\alpha \to R^n \setminus M_\alpha$$

such that: 1) $\varphi_0 = 1$; 2) $\varphi_t|_{R^n \setminus O_\varepsilon(M)} = 1$; 3) $\varphi_1(R^n \setminus M_\alpha) \subset R^n \setminus O_\eta(M)$.

Corollary 1. If $Q = \{q_\alpha\}$ is a class of embeddings close to q, then for every $\varepsilon > 0$ there exist $\delta > 0$ and $\eta > 0$ such that, if q_α is a δ-close-to-q embedding and V is a neighborhood of M_α, then there is a neighborhood $W(M_\alpha)$ such that for every $(n-3)$-dimensional polyhedron P in $O_\eta(M)$ there is an ε-isotopy $h_t : R^n \to R^n$ such that: 1) $h_0 = 1$; 2) $h_t = 1$ on $W \cup (R^n \setminus O_\varepsilon(M))$; 3) $h_1 V \supset P$.

Corollary 2. If q is a locally homotopically unknotted embedding and $Q = \{q_\alpha\}$ is a class of locally homotopically unknotted embeddings close to q, then for every $\varepsilon > 0$ there exist $\eta > 0$ and $\delta > 0$ such that, if $q_\alpha \in Q$ is a δ-close embedding, then for every $(n-3)$-dimensional polyhedron P in $R^n \setminus M_\alpha$ there is an ε-isotopy $h_t : R^n \to R^n$ such that: 1) $h_0 = 1$; 2) $h_t = 1$ on $(R^n \setminus O_\varepsilon(M)) \cup M_\alpha$; 3) $h_1(R^n \setminus O_\eta(M)) \supset P$.

Corollary 1 follows from the definition, and Corollary 2 from the lemma, by the usual method of proof of Stallings' engulfing lemma; moreover, in the case $\dim P = n-3$ one must apply Zeeman' s "The piping lemma" (^{4}). In addition, estimates are needed, which are made in the same way as in the analogous argument in (^{1}).

Main theorem. If $q : M \to R^n$, $n \geqslant 5$, is a locally homotopically unknotted embedding of a compact manifold and $Q = \{q_\alpha\}$ is a class, close to q, of locally homotopically unknotted embeddings, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, if $q_\alpha \in Q$ is a δ-close-to-q embedding and V is a neighborhood of M_α, then there is an ε-isotopy $h_t : R^n \to R^n$ such that: 1) $h_0 = 1$; 2) $h_t = 1$ on $M_\alpha \cup (R^n \setminus O_\varepsilon(M))$; 3) $h_1 V \supset M$.

The proof is based on the standard application of the engulfing lemma (cf. (^{2}), and also (^{5})).

Lemma 2. If $k \neq n-2$ and $q : \widetilde{M} \to R^n$ is an arbitrary embedding, then all locally 1-connected embeddings of $\widetilde{M}$ in R^n form a class of embeddings close to q.

Lemma 3. If $k = n-2$ and $q : \widetilde{M} \to R^n$ is a locally homotopically unknotted embedding, then all embeddings $q_\alpha : \widetilde{M} \to R^n$ for which

$$q_\alpha \widetilde{M} \subset \widetilde{M}$$

and $q_\alpha \widetilde{M}$ is locally flat in M form a class of embeddings close to q.

The argument by means of which Bryant and Seebeck applied their theorem remains valid, in view of these two lemmas, also in our situation. This gives:

Corollary 3. If $q : \widetilde{M} \to R^n$ is an embedding of a manifold, locally flat at interior points and locally 1-connected at boundary points, then it is locally flat also at boundary points.

This proposition leads to the following final formulation of the main result from (^{5}):

Cellular-union theorem. If $q : B^k \to R^n$ is a locally homotopically unknotted embedding which is locally flat at the points of $B^k \setminus B^{k-1}$, then it is locally flat.

By an easy induction one obtains from this (cf. (^{6})):

Corollary 4. If a polyhedron P is embedded in a manifold $M \subset R^n$, which is locally homotopically unknotted at all points and locally flat at the points of $M \setminus P$, in such a way that in some triangulation of P every open simplex is embedded locally flatly in M, and moreover $\dim P < \dim M$, then M is locally flat at all points.

2. Sketch of the proofs of Lemmas 1, 2, 3. We first prove Lemmas 2 and 3. Their proofs are combined in one argument if we denote by $Q = \{q_\alpha\}$ the totality of all locally 1-connected embeddings in Lemma 2 and embeddings into $\widetilde{M}$ in Lemma 3.

(A). If $x \in \dot{M}$, then for i, $0 \leq i \leq n$, and for ε, $0 < \varepsilon < \varepsilon_x$, there exist $\eta > 0$ and $\delta > 0$ such that, if $q_\alpha \in Q$ is a δ-close embedding to q and $S^{n-k-1} \subset O_\eta(x) \setminus M_\alpha$ is a sphere simply linked with M_α in $O_\varepsilon(x)$, then every map $g : B^i \to O_\eta(x) \setminus M_\alpha$ for which $g\dot{B}^i \subset S^{n-k-1}$ is deformed in $O_\varepsilon(x) \setminus M_\alpha$ into S^{n-k-1}, with the boundary fixed.

If $\eta > 0$ and $\delta > 0$ are chosen so that, for some ball neighborhood $H \subset M$,

$$q_\alpha H \cap O_\eta(x) = M_\alpha \cap O_\eta(x)$$

and

$$q_\alpha H \subset O_\varepsilon(x)$$

for all embeddings $q_\alpha \in Q$ that are δ-close to q, then, for $i \neq n-k-1$, all i-dimensional cycles in $O_\eta(x) \setminus M_\alpha$ bound in $O_\varepsilon(x) \setminus M_\alpha$, while the group of $(n-k-1)$-dimensional cycles in $O_\eta(x) \setminus M_\alpha$ is Z. Hence it follows that if in $O_\eta(x) \setminus M_\alpha$ there is a cycle mod S^{n-k-1}, where S^{n-k-1} is simply linked with M in $O_\varepsilon(x)$, then it bounds mod S^{n-k-1} in $O_\varepsilon(x) \setminus M_\alpha$. The proposition is obtained by induction on the cells, as in the proof of the relative Hurewicz theorem, under the condition that it is true for one-dimensional cells. In the one-dimensional case we join the ends of the given path by a path in S^{n-k-1}, and, in the case $k = n-2$, in such a way that the resulting closed contour does not link

$$M_\alpha \cap O_\varepsilon(x) = M \cap O_\varepsilon(x)$$

in $O_\varepsilon(x)$. Then for $k = n-2$ the assertion follows from the local unknottedness of the embedding. For $k \neq n-2$ we contract the contour to a point in $O_\eta(x)$ and then remove this homotopy from M_α in $O_\varepsilon(x)$, as is done in ([3],[7]) and also in ([1]).

The following proposition is proved similarly:

(A′). If $x \in \dot{M}$, then for every $\varepsilon > 0$ there exist $\delta > 0$ and $\eta > 0$ such that, for every embedding $q_\alpha \in Q$ that is δ-close to q, every map of an i-dimensional sphere, $0 \leq i \leq n$, in $O_\eta(x) \setminus M_\alpha$ is homotopic to zero in $O_\varepsilon(x) \setminus M_\alpha$.

Corollary 5. If $x \in M$, then for $\varepsilon > 0$ there exist $\eta > 0$ and $\delta > 0$ such that, for every embedding $q_\alpha \in Q$ that is δ-close to q, the set

$$O_\eta(x) \setminus M_\alpha$$

is deformed in $O_\varepsilon(x) \setminus M_\alpha$ either to a point, if $x \in \dot{M}$, or to an arbitrary sphere

$$S^{n-k-1} \subset O_\eta(x)$$

simply linked with M_α in $O_\varepsilon(x)$, when $x \in \mathring{M}$.

(B). If $x \in \mathring{M}$, then for every ε, $0 < \varepsilon < \varepsilon_x$, there exist $\eta > 0$ and $\delta > 0$ such that, if an embedding $q_\alpha \in Q$ is δ-close to q and

$$y \in M_\alpha \cap O_\eta(x),$$

then for every neighborhood $V(y)$ there is a sphere

$$S^{n-k-1},$$

embedded in

$$(V \setminus M_\alpha) \cap O_\eta(x),$$

simply linked with M_α in $O_\varepsilon(x)$.

Let $\eta > 0$ and $\delta > 0$ be chosen with respect to $\varepsilon > 0$ as in (A). There is a map of the sphere S^{n-k-1} into

$$(V \setminus M_\alpha) \cap O_\eta(x),$$

representing a cycle simply linked with M_α in $O_\varepsilon(x)$. This is clear for $k = n-1$. For $k = n-2$ it follows from the fact that every one-dimensional cycle is represented by a closed contour. For $k \le n-3$ this follows from the local simple connectedness of the embedding and the local form of the Hurewicz theorem. If $k > n/2-1$, then by putting the map in general position we transform it into an embedding. If $k < n/2-1$, then, according to (^{1}), M_α is locally flat and the assertion is trivial. If $k = n/2-1$, then there are zero-dimensional self-intersections, which are removed (if $k \ge 2$, i.e. $n \ge 6$) by Whitney' s method (see (^{8})), which is applicable here in view of the local simple connectedness of the embedding.

(C). If $x \in M$, then for every $\varepsilon > 0$ and for i, $0 \le i \le n$, there exist $\delta > 0$ and $\eta > 0$ such that, for every embedding $q_\alpha \in Q$ that is δ-close to q, for the point

$$y = M_\alpha \cap O_\eta(x)$$

and for every $\rho > 0$ there is $\tau > 0$ such that, if a map

$$g : B^i \to O_\eta(x) \setminus \dot{M}_\alpha$$

is given, with

$$gB^i \subset O_\tau(y),$$

then there exists a homotopy

$$\varphi_t : B^i \to O_\varepsilon(x) \setminus M_\alpha$$

such that: 1) $\varphi_0 = g$; 2) $\varphi_1|\dot{B}^i = 1$; 3) $\varphi_1 B^i \subset O_\rho(y)$.

Choose η and δ with respect to ε, according to (A) or (A′). Then, according to Corollary 5, for the embedding q_α, for

$$y \in M_\alpha \cap O_\eta(x)$$

and for the number

$$\varepsilon' = \min(\rho, d(R^n \setminus O_\eta(x)))$$

we find $\tau > 0$ such that

$$O_\tau(y) \setminus M_\alpha$$

is deformed in

$$O_{\varepsilon'}(y) \setminus M_\alpha$$

to some sphere simply linked with M_α in $O_\varepsilon(x)$ (see (B)). The homotopy φ_t has thereby already been defined on

$$B^i \times 0$$

and

$$\dot{B}^i \times [0,1].$$

According to

by the choice of τ it can be extended to $(B^i \setminus \dot{B}_1^i) \times 1$, where B_1^i is a concentric smaller ball, so that

$$\varphi_1(B^i \setminus \dot{B}_1^i) \subset O_\rho(y) \setminus M_\alpha, \qquad \varphi_1 B_1^i \subset S^{n-k-1}.$$

We obtain a mapping of the cell into $O_\eta(x) \setminus M_\alpha$ such that its boundary lies on S^{n-k-1}, and, according to the choice of η and δ, we can extend φ_t to all of $B^i \times [0, 1]$, as required.

With the help of a compactness argument we obtain:

(D). For every $\varepsilon > 0$ and for $i,\ 0 \le i \le n$, there exist $\delta > 0$ and $\eta > 0$ such that, for an embedding $q_\alpha \in Q$ that is δ-close to q, and for $\rho > 0$, there is a $\tau > 0$ such that, if a mapping $g : B^i \to R^n \setminus M_\alpha$ is given such that the diameter of gB^i is less than η, and $g\dot{B}^i$ lies in the τ-neighborhood of a point of M_α, then there exists an ε-homotopy

$$\varphi_t : B^i \to R^n \setminus M_\alpha$$

such that $\varphi_t|_{\dot{B}^i} = 1$, and $\varphi_1 B^i$ lies in the ρ-neighborhood of a point of M_α.

Hence our assertion follows by induction over the skeleta of a sufficiently fine triangulation of a sufficiently close neighborhood of M.

Let us pass to Lemma 1.

(E). If $x \in \dot{M}$, then for every $\varepsilon < \varepsilon_x$ one can find $\eta > 0$ such that, if the sphere $S^{n-k-1} \subset O_\eta(x)$ links M once in $O_\varepsilon(x)$, then there exists $\delta > 0$ such that, if the embedding $q_\alpha \in Q$ is δ-close to q and a singular cell is given in $O_\eta(x) \setminus M_\alpha$ with boundary on S^{n-k-1}, then it is deformed, in $O_\varepsilon(x) \setminus M_\alpha$, into S^{n-k-1} with the boundary fixed.

If $k \ne n-2$, then by the Hurewicz theorem the assertion is reduced to the case of a one-dimensional cell, where the proof is the same as in (A). If $k = n-2$, then, since q_α is a class of embeddings close to q, we easily reduce the matter to the case where the cell together with the sphere lies arbitrarily close to M_α, and then the assertion follows from the local homotopic unlinking of M_α.

(F). If $x \in M$, then, for $\varepsilon > 0$ and for $i,\ 0 \le i \le n$, there exists $\eta > 0$ such that for every $\rho > 0$ one can find $\tau > 0$ and $\delta > 0$ so that, if $q_\alpha \in Q$ is δ-close to q and we are given a singular i-cell in $O_\eta(x) \setminus M_\alpha$, whose boundary lies in $R^n \setminus O_\rho(M)$, then there is a deformation of this cell in $O_\varepsilon(x) \setminus M_\alpha$ into $R^n \setminus O_\tau(M)$, with the boundary fixed.

Let, for example, $x \in \dot{M}$ and $\varepsilon < \varepsilon_x$. Since M is locally homotopically unknotted, it follows from Corollary 5 that, for some $\eta > 0$, there is a deformation of $O_\eta(x) \setminus M$ in $O_\varepsilon(x) \setminus M$ into some sphere

$$S^{n-k-1} \subset O_\eta(x) \setminus M.$$

For a given $\rho > 0$, choose $\tau > 0$ so that during the deformation the image of $O_\eta(x) \setminus O_\rho(M)$ does not meet $O_\tau(M)$. If now an i-cell is given as in the condition, then, by the choice of τ, we may assume that its boundary already lies in S^{n-k-1}, and then see (E).

With the help of a compactness argument we obtain from this:

(G). For $\varepsilon > 0$ and $i,\ 0 \leq i \leq n$, there exists $\eta > 0$ such that, for every $\rho > 0$, one can find $\delta > 0$ and $\tau > 0$ so that, if an embedding $q_\alpha \in Q$ is δ-close to q and we are given a singular i-cell in the η-neighborhood of some point of M outside M_α such that its boundary lies in $R^n \setminus O_\rho(M)$, then there is an ε-deformation of this cell in $R^n \setminus M_\alpha$ into $R^n \setminus O_\tau(M)$, with the boundary fixed.

From (G), Lemma 1 is obtained by the usual induction over the skeleta of a sufficiently fine triangulation of a sufficiently close neighborhood of M, from which M_α has been removed.

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