ON BOUNDARY-VALUE PROBLEMS OF LINEAR ELECTRODYNAMICS OF PLASMA

MATHEMATICAL PHYSICS

1968

SovietRxiv

Abstract

Full Text

UDC 533.9

MATHEMATICAL PHYSICS

Yu. V. DNESTROVSKII, D. P. KOSTOMAROV

ON BOUNDARY-VALUE PROBLEMS OF LINEAR ELECTRODYNAMICS OF PLASMA

(Presented by Academician A. N. Tikhonov, 26 VI 1967)

1. In plasma, unlike ordinary media, the material equations relating the induction **D** and the current **j** to the electric field **E** have not a pointwise but an integral character. As a result, Maxwell' s equations lead to integro-differential equations for the components of the electric field. Such a change in the mathematical nature of the equations raises the question of the character of the additional conditions that must be prescribed for a unique determination of the solution.

In the present paper we consider a system of integro-differential equations describing the propagation of electromagnetic waves in a half-bounded magnetoactive plasma for an arbitrary orientation of the external magnetic field $\mathbf{H}_0$ with respect to the boundary. It is proved that its solution, as in ordinary electrodynamics, is uniquely determined if the initial values of the vectors **E** and **H** and the boundary values of the tangential components of the vector **E** are prescribed. Such a result is quite natural (see, for example, ([1])), but it is not obvious; therefore it requires a rigorous mathematical justification.

The essence of the matter may be explained as follows. In media without spatial dispersion, at a given oscillation frequency there may exist two normal waves of different polarization. Their amplitudes are uniquely determined by prescribing the two tangential components of the electric field at the boundary. For a plasma, the dispersion equation may have an infinite set of roots $k(\omega)$, i.e., the number of normal waves is infinitely large. However, the investigation of the formulation of the problem carried out by us has shown that the determination of their amplitudes does not require an increase in the number of boundary conditions. This result makes it possible, in particular, to justify the continuation method, which has been used by many authors for solving boundary-value problems in the case of a half-bounded plasma and was recently called into question in ([2]).

Sometimes, in order to simplify problems of this type, the integral current operator is expanded, approximately replacing it by a differential operator. In this case the order of the equation may increase, and then the boundary conditions become insufficient. However, the indeterminacy in this case is connected not with the problem under consideration itself, but with the transformations made in it. Therefore the additional boundary conditions must be derived from the original integro-differential equations together with the derivation of the approximate differential equations, as was done, for example, in ([3]).

2. Let the half-space $x > 0$ be filled with a magnetoactive plasma with immobile ions and Maxwellian electrons, and suppose that at the plasma boundary the condition of specular or diffuse reflection is satisfied for the electrons. Consider the system of equations describing the propagation of electromagnetic waves in such a plasma in the direction of the x-axis $(\mathbf{E} = \mathbf{E}(t, x),\ \mathbf{H} = \mathbf{H}(t, x))$:

$$\operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}, \qquad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}. \tag{1}$$

Here $\mathbf{j}(t, x)$ is the current due to the motion of the electrons. Solving the linea-

linearized Vlasov kinetic equation, one can obtain for it the following expression:

$$\mathbf{j}(t, x) = \frac{\omega_0^2}{4\pi}\int \frac{df_0}{dv}\frac{\mathbf{v}}{v}\, d^3v \int_{t-T(x,\mathbf{v})}^{t} \mathbf{E}(\tau, \xi(t-\tau, x, \mathbf{v}))\, \mathbf{u}(t-\tau, x, \mathbf{v})\, d\tau, \tag{2}$$

where $f_0 = f_0(v)$ is the Maxwellian distribution; ω_0 is the plasma frequency; $T(x, \mathbf{v})$, $\xi(t-\tau, x, \mathbf{v})$, $\mathbf{u}(t-\tau, x, \mathbf{v})$ are functions characterizing the unperturbed motion of electrons in the half-space $x > 0$ in the presence of an external magnetic field $\mathbf{H}_0$. In the case of a specularly reflecting boundary, to determine them one must solve the following problem:

$$\frac{d\vec{\rho}}{d\tau} = \mathbf{u}, \qquad \frac{d\mathbf{u}}{d\tau} = \frac{e}{mc}[\mathbf{u} \times \mathbf{H}_0] \quad (0 \leqslant \tau \leqslant t), \qquad \vec{\rho}\big|_{\tau=t} = \mathbf{r}, \quad \mathbf{u}\big|_{\tau=t} = \mathbf{v}, \tag{3}$$

where the function ξ that interests us is the projection of the vector $\vec{\rho}$ onto the x-axis. If the trajectory reaches the plasma boundary, then the normal component of the velocity u_x must change sign (specular reflection). The subsequent course of the trajectory is again determined by equations (3). The lower limit of integration in the integral with respect to τ is equal to zero (i.e., $T(x, \mathbf{v}) = t$). In the case of a diffusely reflecting boundary, the functions ξ and $\mathbf{u}$ are also determined with the aid of the solution of problem (3), while the function $T(x, \mathbf{v})$ is chosen as follows. For those trajectories which during the time $0 \leqslant \tau \leqslant t$ do not reach the boundary, $T(x, \mathbf{v}) = t$. For the remaining trajectories, $T(x, \mathbf{v})$ is the time τ at which they first reach the boundary. We note that, by virtue of equations (3), the absolute value of the velocity u is an integral of motion: $u(t - \tau, x, \mathbf{v}) = v$, and consequently $|u_\alpha| \leqslant v$ $(\alpha = x, y, z)$.

3. Let us consider equations (1), (2) under the prescribed boundary and initial conditions:

$$
\begin{aligned}
E_y(t,0) &= \mu_y(t), \qquad E_z(t,0) = \mu_z(t),\\
\mathbf{E}(0,x) &= \mathbf{a}(x), \qquad \mathbf{H}(0,x) = \mathbf{b}(x),\\
(a_x(x) &= \text{const}, \qquad b_x(x) = 0).
\end{aligned}
\tag{4}
$$

Existence and uniqueness theorem. Let the functions $\mu_y(t)$, $\mu_z(t)$, $\mathbf{a}(x)$, $\mathbf{b}(x)$ be bounded and continuously differentiable; then problem (1), (2), (4) has a unique solution in the class of functions bounded in any domain of the form $0 \leqslant t \leqslant t_0,\ 0 \leqslant x < \infty$.

Under the assumptions made regarding the initial and boundary conditions, the problem under consideration is equivalent to a system of integral equations of Volterra type

$$E_\alpha(t,x) = I_\alpha[j_\alpha] + F_\alpha(t,x) \qquad (\alpha = x, y, z). \tag{5}$$

Here

$$I_x[j_x] = -4\pi \int_0^t j_x(t',x)\,dt', \qquad F_x(t,x) = a_x;$$

$$I_{y,z}[j_{y,z}] = -2\pi \int_0^t \{j_{y,z}(t', x + c(t-t')) + j_{y,z}(t', x - c(t-t'))\}\,dt',$$

$$F_{y,z}(t,x) = \frac{1}{2}[a_{y,z}(x+ct) + a_{y,z}(x-ct)] \mp$$

$$\mp \frac{1}{2}[b_{z,y}(x+ct) + b_{z,y}(x-ct)]$$

for $0 < t < x/c$;

$$I_{y,z}[j_{y,z}] = -2\pi \left\{ \int_0^t j_{y,z}(t', x + c(t-t'))\,dt' - \int_0^{t-x/c} j_{y,z}(t', -x + c(t-t'))\,dt' + \int_{t-x/c}^t j_{y,z}(t', x + c(t-t'))$$

$$F_{y,z}(t,x) = \left[\mu_{y,z}(t - x/c) + \frac{1}{2} a_{y,z}(x+ct)\right] - a_{y,z}(ct - x) \mp \frac{1}{2}\left[b_{z,y}(x+ct) + b_{z,y}(ct-x)\right]$$

for $0 < x/c < t$.

Taking into account expression (2) for the current, the operators I_α are, in the final analysis, integral operators depending on the components of the electric field.

Applying the method of successive approximations to system (5), it is easy to prove the existence of a solution. Analysis of the corresponding homogeneous system shows that it admits only the trivial solution, whence uniqueness follows.

4. In conclusion, let us consider the boundary-value problem with boundary conditions harmonically dependent on time:

$$E_{y,z}(t,0) = E^0_{y,z} e^{-i\omega t}, \tag{6}$$

We shall seek its solution in the form $E(t,x) = E(x)e^{-i\omega t}$, requiring that this function represent the limiting form, as $t \to \infty$, of the solution of the problem with zero initial conditions and boundary conditions (6). According to the theorem proved, the solution of such a problem on steady-state oscillations must be unique.

Moscow State University
named after M. V. Lomonosov

Received
30 V 1967

CITED LITERATURE

1. V. P. Silin, A. A. Rukhadze, *Electromagnetic Properties of Plasma and Plasma-like Media*, 1961.

2. C. Comstock, Phys. Fluids, **9**, 1514 (1966).

3. V. P. Silin, E. P. Fetisov, ZhETF, **41**, 159 (1961).

Note: Figure translations are in progress. See original paper for figures.