ON THE ORDER OF THE NEIGHBORHOOD OF MAXIMAL INTERVALS FOR ALMOST ALL FUNCTIONS OF THE ALGEBRA OF LOGIC

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 519.95

MATHEMATICS

A. A. SAPOZHENKO

ON THE ORDER OF THE NEIGHBORHOOD OF MAXIMAL INTERVALS FOR ALMOST ALL FUNCTIONS OF THE ALGEBRA OF LOGIC

(Presented by Academician M. A. Lavrentiev on 16 VI 1967)

In papers (^{1,2}) the definition of a local algorithm for computing information was formulated. In this note we estimate the order of the principal neighborhood of a maximal interval (^{2}). The magnitude of this parameter characterizes the complexity of the algorithm in the minimization of Boolean functions. It will be shown that, for almost all functions, the order of the neighborhood of any interval of nonzero dimension is asymptotically equal to $n \log_2 \log_2 n$.

Let P_n denote the set of all functions of the algebra of logic in n arguments. If $f \in P_n$, then N_f is the set of vertices α of the n-dimensional cube E^n such that $f(\alpha) = 1$, and $\mathfrak{M}_f$ is the set of all maximal intervals of the function f (see (^{3})). We shall regard the n-dimensional cube E^n as a graph whose vertices are the strings of zeros and ones of length n, and whose edges are pairs of strings differing in exactly one coordinate. By $G(f)$ we denote the subgraph (see (^{4})) of the cube E^n generated by the set N_f, and by $\Gamma(f)$ the graph whose vertices are the maximal intervals from $\mathfrak{M}_f$, and whose edges are pairs of maximal intervals intersecting as faces of the cube E^n (see (^{3})).

Let H be some graph. An ordered set of edges of the graph $H[(\alpha, \beta_1), (\beta_1, \beta_2), \dots, (\beta_{l-1}, \gamma)]$ will be called a chain connecting the vertices α and γ. We shall denote this chain by the symbol $[\alpha, \beta_1, \dots, \beta_{l-1}, \gamma]$ or $[\alpha, \gamma]$. The length $l_{[\alpha,\gamma]}$ of the chain $[\alpha, \gamma]$ will be the number of edges of this chain. The distance between vertices α and β of the graph H will be the length of the shortest chain connecting these vertices, and will be denoted by $\rho_H(\alpha, \beta)$. If H is disconnected and the vertices α, β lie in different components, then the distance $\rho_H(\alpha, \beta)$ is not defined. Denote by $R(H)$ and $D(H)$, respectively, the radius and diameter of a connected graph H (see (^{4})). The radius (diameter) of a disconnected graph will be the maximum of the radii (diameters) of its components.

Definition 1 (see (^{2})). Let $f \in P_n$. The principal neighborhood of order zero of a maximal interval M from $\mathfrak{M}_f$ is the set $\mathcal{E}_0(M, \mathfrak{M}_f)$, consisting of the interval

M itself.

The principal neighborhood of order k is the set $\mathcal{E}_k(M, \mathfrak{M}_f)$ of intervals L from $\mathfrak{M}_f$ satisfying one of the following conditions:

1°. There exists a maximal interval K from $\mathcal{E}_{k-1}(M, \mathfrak{M}_f)$ and a vertex α of the cube E^n such that $\alpha \in K \cap L$.

2°. There exist intervals $K_1, K_2, \dots, K_s$ satisfying 1°, such that every vertex α of the interval L belongs to at least one of these intervals.

The least number $\pi(M) = \pi(M, \mathfrak{M}_f)$ such that, for all $i > \pi(M)$, $\mathcal{E}_{i+1}(M, \mathfrak{M}_f) = \mathcal{E}_i(M, \mathfrak{M}_f)$, will be called the largest order of the principal neighborhood of the maximal interval M.

We shall say that almost all functions of the algebra of logic possess property A if

$$\lim_{n\to\infty} \Phi^n(A)/2^{2^n} = 1.$$

Here $\Phi^n(A)$ –

the number of functions from P_n possessing property A. For example, it is well known that almost all functions of the algebra of logic satisfy the condition*

$$2^{n-1} - \sqrt{n}\,2^{n/2} \le |N_f| \le 2^{n-1} - \sqrt{n}\,2^{n/2}.$$

In (5) it is proved that almost all functions of the algebra of logic have the following properties:

1°. The graph $G(f)$ consists of no more than $\eta(n)$ components, where $\eta(n)$ is an arbitrary function such that $\lim\limits_{n\to\infty} \eta(n) = \infty$ and $\eta(n) \le n/2\log_2 n$. One of the components of the graph $G(f)$, which we shall call the principal component and denote by K_1, satisfies the condition

$$|N_f| - \eta(n) \le |K_1| \le |N_f|.$$

Roughly speaking, this means that the principal component contains almost all vertices of the graph $G(f)$. The remaining components, if there are any, are isolated vertices of the graph $G(f)$.

2°. $n-2 \le R(G(f)) \le n-1, \quad D(G(f)) = n+1.$

3°. If α, β are nonisolated vertices of the graph $G(f)$ and $\rho_{E^n}(\alpha,\beta) < n/\log_2 n$, then

$$\rho_{G(f)}(\alpha,\beta) \le \rho_{E^n}(\alpha,\beta) + 4.$$

4°. By the sphere $S_l^n(\alpha)$ of radius l in the cube E^n we mean the set of vertices of the cube lying at distance l from α in the metric E^n. The graph $G(f)$ is arranged so that, whatever sphere $S_l^n(\alpha)$ one takes such that $2 \le l \le n-2$, there exists a vertex β of the principal component of the graph $G(f)$ belonging to the sphere $S_l^n(\alpha)$.

It follows from property 1° that, for almost all functions of the algebra of logic, the graph $\Gamma(f)$ consists of no more than $\eta(n)$ connected components. One of the components of the graph $\Gamma(f)$ contains at least $\mathfrak{M}_f - \eta(n)$ maximal intervals, while the remaining ones are maximal intervals of zero dimension. By the dimension of an interval we mean its dimension as a face of the cube E^n.

It is also clear that for a maximal interval M of zero dimension $\pi(M) = 0$. For maximal intervals of nonzero dimension, the inequalities

$$R(\Gamma(f)) \leq \pi(M) \leq D(\Gamma(f)) + 1$$

are evidently valid.

Theorem. For almost all functions of the algebra of logic, for every maximal interval M of nonzero dimension,

$$\pi(M) \sim n/\log_2 \log_2 n.$$

For the proof of the theorem it is enough to verify the validity of the following two assertions.

Assertion 1. For almost all functions of the algebra of logic

$$R(\Gamma(f)) \geq (n-2)/[\log_2 \log_2 n + \log_2 \log_2 \log_2 n + 1].$$

Assertion 2. For almost all functions of the algebra of logic

$$D(\Gamma(f)) \leq \frac{n}{\log_2 \log_2 n}(1 + O(1)).$$

Let α, β be vertices of the cube E^n, and let $\rho_{E^n}(\alpha, \beta) = k$. The set $E^k(\alpha, \beta)$ of vertices γ satisfying the condition

$$\rho_{E^n}(\alpha, \gamma) + \rho_{E^n}(\gamma, \beta) = \rho_{E^n}(\alpha, \beta)$$

is, obviously, a k-dimensional face of the cube (see (3)). The subgraph of the cube E^n generated by the set $E^k(\alpha, \beta)$ will be called the k-dimensional subcube spanned by the vertices α, β, and will be denoted by the same symbol $E^k(\alpha, \beta)$.**

---

* If Q is some set, then by the symbol $|Q|$ we shall denote the number of elements in Q.

** Everywhere, a subset of vertices of the n-dimensional cube and the subgraph generated by this set will be denoted by the same symbol. It will be clear from the context which of the concepts is meant.

Definition 2. Let $\gamma_0, \gamma_1, \dots, \gamma_q$ be a sequence of pairwise distinct vertices of the cube E^n, $d_i = \rho_{E^n}(\gamma_{i-1}, \gamma_i)$, $\sum_{i=1}^{q} d_i = m$, and let $E^{d_i}(\gamma_{i-1}, \gamma_i)$ be the subcube spanned by the vertices γ_{i-1}, γ_i. We shall denote by

$$\mathfrak{z}_m(\gamma_0, \gamma_1, \dots, \gamma_q)$$

or, if this causes no misunderstanding, by

$$\mathfrak{z}_m(\gamma_0, \gamma_{i_1}, \dots, \gamma_{i_k}, \gamma_q),$$

where some of the γ_i are omitted, the set

$$\bigcup_{i=1}^{q} E^{d_i}(\gamma_{i-1}, \gamma_i).$$

We shall call this set an (m, q)-chain connecting the vertices γ_0, γ_q, if two conditions are satisfied: 1)

$$E^{d_i}(\gamma_{i-1}, \gamma_i) \cap E^{d_{i+1}}(\gamma_i, \gamma_{i+1}) = \{\gamma_i\}$$

for all $i = 1, 2, \dots, q-1$; 2)

$$E^{d_i}(\gamma_{i-1}, \gamma_i) \cap E^{d_j}(\gamma_{j-1}, \gamma_j) = \emptyset$$

if $|i - j| > 1$. We shall call the vertices γ_0, γ_q the **ends**, and the vertices $\gamma_1, \gamma_2, \dots, \gamma_{q-1}$ the **nodal vertices** of the (m, q)-chain $\mathfrak{z}_m(\gamma_0, \gamma_q)$. If $f \in P_n$ and $\mathfrak{z}_m(\gamma_0, \gamma_q) \subseteq N_f$, then we shall say that the set N_f contains the chain $\mathfrak{z}_m(\gamma_0, \gamma_q)$, or that $\mathfrak{z}_m(\gamma_0, \gamma_q)$ is an (m, q)-chain of the function f.

Let us note the connection between the chains defined earlier in the graphs $G(f)$, $\Gamma(f)$, and (m, q)-chains. Obviously, every non-self-intersecting chain $[\alpha, \beta]$ of length m in the graph $G(f)$ is an (m, m)-chain. If $[M_1, M_{q+1}]$ is a shortest chain in $\Gamma(f)$ connecting the maximal intervals M_1, M_{q+1}, then for every pair of vertices α, β such that $\alpha \in M_1$, $\beta \in M_{q+1}$, there exists an (m, q')-chain $\mathfrak{z}_m(\alpha, \beta)$ ($m = \rho_{E^n}(\alpha, \beta)$, $q' \le q - 1$) such that every vertex of the chain $\mathfrak{z}(\alpha, \beta)$ belongs to the chain $[M_1, M_{q+1}]$. On the other hand, for every (m, q)-chain $\mathfrak{z}(\alpha, \beta)$ there exists a chain $[M_1, M_{q'}]$ such that $q' \le q$, $\alpha \in M_1$, $\beta \in M_{q'}$, and every vertex of the chain $\mathfrak{z}(\alpha, \beta)$ belongs to the chain $[M_1, M_{q'}]$. If the set N_f contains an (m, q)-chain $\mathfrak{z}(\alpha, \beta)$ and $m > q$, then for every q' ($q \le q' \le m$) there exist (m, q')-chains connecting α, β.

Lemma 1. Let m, n be integers, $m \ge n - 2$,

$$q_0 = q_0(n) = [(n-2)/(\log_2 \log_2 n + \log_2 \log_2 \log_2 n + 1)].$$

Denote by $\Phi^n(m, q_0)$ the number of functions f from P_n such that N_f contains an (m, q_0)-chain, and let

$$\Phi^n(q_0) = \sum_{m \ge n-2} \Phi^n(m, q_0).$$

Then

$$\lim_{n\to\infty} \Phi^n(q_0)/2^{2^n} = 0.$$

Let $\mathfrak{K}_1(n)$ be the class of functions from P_n possessing the following properties:

1°. The set N_f contains not a single (m, q_0)-chain, $m \geq n-2$,

$$q_0 = [(n-2)/(\log_2 \log_2 n + \log_2 \log_2 \log_2 n + 1)].$$

2°. The graph $G(f)$ consists of no more than $\eta(n)$ components, $\eta(n) < n/\log_2 n$. There exists a principal component K_1, containing not fewer than $|N_f| - \eta(n)$ vertices from N_f; all other components are isolated points, and $|N_f| \sim 2^{n-1}$.

3°. For every vertex α from K_1 there exists a vertex $\beta \in K_1$ such that

$$\rho_{E^n}(\alpha, \beta) = n-2.$$

From the preceding it is clear that

$$\lim_{n\to\infty} |\mathfrak{K}_1(n)|/2^{2^n} = 1.$$

Proof of Assertion 1. We shall show that for every function $f \in \mathfrak{K}_1(n)$ and any of its maximal intervals M of nonzero dimension there exists a maximal interval M' such that

$$\rho_{\Gamma(f)}(M, M') \geq q_0.$$

Thereby Assertion 1 will be proved.

Let $f \in \mathfrak{K}_1(n)$, $M \in \mathfrak{M}_f$, and let the dimension of M be greater than 0. Let $\alpha \in M$. By properties 2°, 3° of functions from the class $\mathfrak{K}_1(n)$, $\alpha \in K_1$, and there exists a vertex $\beta \in K_1$ such that $\rho_{E^n}(\alpha, \beta) = n-2$. Let M' be an arbitrary maximal interval from $\mathfrak{M}_f$ such that $\beta \in M'$. We shall show that

$$\rho_{\Gamma(f)}(M, M') \geq q_0.$$

Suppose the contrary. Then there exists a chain $[M, M']$ of the graph $\Gamma(f)$ of length $q < q_0$, and hence also an (m, q_0)-chain $\mathfrak{z}(\alpha, \beta)$ such that $m \geq n-2$, which contradicts property 1° of functions from $\mathfrak{K}_1(n)$. Assertion 1 is proved.

Definition 3. Let $f \in P_n$, $\alpha, \beta \in N_f$, $\rho_{E^n}(\alpha, \beta) = K$. The pair (α, β) will be called a bad k-pair of the function f if α, β are not isolated and N_f contains no (m, q_1)-chain connecting α with β such that

$$k \leq m \leq k+12, \qquad q_1 = q_1(k) \leq [k/(\log_2 \log_2 k - \log_2 \log_2 \log_2 k)] + 12.$$

Lemma 2. Let $\mathfrak{K}_2(n)$ be the number of functions in P_n that have at least one bad k-pair, and let

$$\Phi^n = \sum_{n/\log_2 n \leq k \leq n - \sqrt{n \log_2 n}} \Phi^n(k).$$

Then

$$\lim_{n\to\infty} \Phi^n/2^{2^n} = 0.$$

Let $\mathfrak{K}_2(n)$ be the class of functions f from P_n having the properties:

1^0. For any nonisolated vertices α, β from N_f such that

$$\rho_{E^n}(\alpha,\beta) \le n/\log_2 n, \qquad \rho_{G(f)}(\alpha,\beta) \le \rho_{E^n}(\alpha,\beta) + 4.$$

2^0. In every sphere $S_n^l(\alpha)$ such that $2 \le l \le n-2$, there exists a vertex β belonging to the principal component of the graph $G(f)$.

3^0. There are no bad k-pairs $(n/\log_2 n \le k \le n - \sqrt{n\log_2 n})$. From the preceding it is clear that

$$\lim_{n\to\infty} |\mathfrak{K}_2(n)|/2^{2^n} = 1.$$

Proof of Assertion 2. It suffices to show that, for every function f from $\mathfrak{K}_2(n)$ and any two maximal intervals M and M' of nonzero dimension from $\mathfrak{M}_f$,

$$\rho_{\Gamma(f)}(M,M') \le \frac{n}{\log_2\log_2 n}(1+o(1)).$$

Let $f \in \mathfrak{K}_2(n)$, and let M and M' be maximal intervals of nonzero dimension from $\mathfrak{M}_f$, and let α_0, β_0 be vertices of the set N_f such that $\alpha_0 \in M,\ \beta_0 \in M'$, and

$$\rho_{E^n}(\alpha_0,\beta_0) = \min_{\alpha\in M,\ \beta\in M'} \rho_{E^n}(\alpha,\beta).$$

Three cases may occur:

a) $\rho_{E^n}(\alpha_0,\beta_0) < n/\log_2 n$;

b) $n/\log_2 n \le \rho_{E^n}(\alpha_0,\beta_0) \le n - \sqrt{n\log_2 n}$;

c) $n - \sqrt{n\log_2 n} \le \rho_{E^n}(\alpha_0,\beta_0) \le n$.

Case a). By property 1^0 of the function f from $\mathfrak{K}_2(n)$,

$$\rho_{G(f)}(\alpha_0,\beta_0) \le \rho_{E^n}(\alpha_0,\beta_0) + 4 \le n/\log_2 n + 4.$$

Hence it easily follows that

$$\rho_{\Gamma(f)}(M,M') \le n/\log_2 n + 5.$$

Case b). By property 3^0 of the function f from $\mathfrak{K}_2(n)$, there exists an (m, q_1)-chain connecting α_0 with β_0, where

$$m \le k + 12, \qquad q_1 = q_1(k) \le [k/(\log_2\log_2 k - \log_2\log_2\log_2 k)] + 12,$$

where $k = \rho_{E^n}(\alpha_0, \beta_0)$.

Hence it easily follows that

$$\rho_{\Gamma(f)}(M, M') \leq \frac{n}{\log_2 \log_2 n}(1 + o(1)).$$

Case c). By property 2^0 of the function f from $\mathfrak{K}_2(n)$, there exists a vertex γ of the principal component of the graph $G(f)$ such that

$$\rho_{E^n}(\alpha_0, \gamma) =]\sqrt{n \log_2 n}[,$$

$$\rho_{E^n}(\gamma, \beta_0) = \rho_{E^n}(\alpha_0, \beta_0) -]\sqrt{n \log_2 n}[.$$

Since

$$\rho_{E^n}(\alpha_0, \gamma) < n / \log_2 n,$$

by property 1^0, in $G(f)$ there exists a chain $[\alpha_0, \gamma]$ of length

$$\leq]\sqrt{n \log_2 n}[,$$

and by property 3^0 of the function f there exists an (m, q_2)-chain connecting γ with β_0,

$$m \leq k + 12, \qquad q_1 = q_1(k) \leq [k/(\log_2 \log_2 k - \log_2 \log_2 \log_2 k)] + 12,$$

where

$$k = \rho_{E^n}(\gamma, \beta_0).$$

Hence it is clear that

$$\rho_{\Gamma(f)}(M, M') \leq \frac{n}{\log_2 \log_2 n}(1 + o(1)).$$

Assertion 2, and therefore also the theorem, are proved.

The author expresses his gratitude to V. V. Glagolev for a number of useful remarks.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

Received
7 VI 1967

References

1. Yu. I. Zhuravlev, in: *Discrete Analysis*, no. 1, Novosibirsk, 1963, p. 5.

2. Yu. I. Zhuravlev, *Dokl. Akad. Nauk SSSR* **151**, no. 5 (1963).

3. S. V. Yablonskii, *Trudy Mat. Inst. im. V. A. Steklova AN SSSR* **51** (1958).

4. C. Berge, *The Theory of Graphs and Its Applications*, IL, 1962.

5. A. A. Sapozhenko, *Discrete Analysis*, no. 10, Novosibirsk, 1967, p. 12.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.