ASYMPTOTIC PROPERTIES AND APPROXIMATION OF STOCHASTIC MODELS OF LEARNING

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Abstract

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ASYMPTOTIC PROPERTIES AND APPROXIMATION OF STOCHASTIC MODELS OF LEARNING

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Many of the simplest procedures of learning (skill formation) consist of a series of consecutive experiments $E_1, \dots, E_t, \dots$. Each experiment begins with the appearance of a conditional stimulus $C_\alpha \in C$ (C is the set of conditional stimuli) with probability $p_t(C_\alpha)$. If the subject (or learner) responds to the stimulus by a reaction $R_\beta \in R$ (R is the set of reactions), then with probability $p_t(S_\gamma / C_\alpha R_\beta)$ an unconditioned stimulus $S_\gamma \in S$ appears (S is the set of unconditioned stimuli). Everything that the subject can learn from experiment E_t is contained in the event $\omega_i = \{C_\alpha R_\beta S_\gamma\} \in C \times R \times S = \Omega$. The pair of finite sets (C, S) and conditional probabilities $p_t(C_\alpha)$, $p_t(S_\gamma / C_\alpha R_\beta)$ will be called the environment in which the experiments E_t take place.

We make the following three assumptions:

a) In experiment E_t the subject (learner) performs reaction R_β with probability $p_t(R_\beta / C_\alpha)$. The matrix of conditional probabilities $\|p_t(R_\beta / C_\alpha)\|$ will henceforth be called the subject' s behavior and denoted by $\bar{p}_t(\beta/\alpha)$.

b) All the experience accumulated by the subject (learner) is completely contained in its behavior $\bar{p}_t(\beta/\alpha)$. Formally this is expressed as follows: if in experiment E_t the event $\omega_{i_t} \in \Omega$ occurred, then the new behavior will be

$$\bar{p}_{t+1}(\beta/\alpha) = \varphi(\bar{p}_t(\beta/\alpha), \omega_{i_t}),$$

where $\bar{p}_t(\beta/\alpha)$ is the behavior of the subject in E_t.

c) The rule for changing behaviors $\varphi(\bar{p}(\beta/\alpha), \omega)$, defined on the set of all behaviors $\hat{P} = \{\|p(R_\beta / C_\alpha)\|\}$, is a contracting homeomorphic mapping with contraction coefficient a_ω and fixed point $\bar{p}_\omega(\beta/\alpha)$. This fixed point is naturally called the behavior taught by the event ω; a_ω is the learning rate.

Learning according to scheme a)–c) will be called a behavioristic model, or briefly BM (for more on BM see ([8])). A BM for which the transformation

$$\varphi(\bar{p}(\beta/\alpha), \omega) = a_\omega \bar{p}(\beta/\alpha) + (1 - a_\omega)\bar{p}_\omega(\beta/\alpha),$$

i.e., is linear, was proposed and tested in many experiments in ([1–3]). This model will henceforth be denoted MBM (the Bush–Mosteller model). A BM can be represented by an automaton with an uncountable set of states (see the addenda in ([3])).

We shall call an approximation of a BM of order k (briefly ABM k) a learning model possessing properties a), b) and the new property c′):

c′) The rule for changing behaviors

$$\tilde{\varphi}(\bar{p}_{\omega_{i_1} \ldots \omega_{i_k}}(\beta/\alpha), \omega_j) = \bar{p}_{\omega_{i_2} \ldots \omega_{i_k} \omega_j}(\beta/\alpha)$$

is defined on the finite set

$$\bar{P}_k = \{\|\bar{p}_{\omega_{i_1} \ldots \omega_{i_k}}(\beta/\alpha)\|\}$$

of all possible behaviors that are fixed points of k-fold superpositions of the mappings

$$\varphi(\varphi(\varphi \ldots (\varphi(\bar{p}(\beta/\alpha), \omega_{i_1}), \omega_{i_2}), \ldots), \omega_{i_k}), \quad \omega_{i_j} \in \Omega.$$

An ABM k can be represented by a combination of a random-signal generator and a finite automaton. In particular, an ABM k is realizable in the form of a homogeneous network of formal neurons and neurons with spontaneous activity.

Beginning with the second experiment E_2, the behavior of BA and ABM k forms a random sequence $\hat{p}_t(\beta/\alpha)$—a Markov chain. The study of asympt—

···of the asymptotic properties of the BM is reduced to the investigation of the asymptotic properties of the chain $p_t(\beta/\alpha)$.

Consider a Markov chain $\xi(t)$, whose set of states is the complete metric space X with metric $\rho(x, y)$ and

$$\max_{x,y \in X} \rho(x, y) = d < \infty.$$

If at time t, $\xi(t) = x \in X$, then with probability $p_t(\omega_i/\xi(t) = x)$ a contracting homeomorphic mapping $\psi(x, \omega_i)$ is applied to the point x, having contraction coefficient $a_{\omega_i} \leq \alpha < 1$ and fixed point x_{ω_i}. The mapping $\psi(x, \omega_i)$ takes $\xi(t) = x$ from x to

$$\xi(t + 1) = \psi(\xi(t), \omega_i) = \psi(x, \omega_i).$$

In the case of a BM, $X = \hat{P}$,

$$p(\omega_i = (C_\alpha R_\beta S_\gamma)/\xi(t)) = \bar{p}_t(\beta/\alpha) = p_t(C_\alpha)p_t(R_\beta/C_\alpha)p_t(S_\gamma/C_\alpha R_\beta).$$

Denote by: Ω^t the set of all sequences $(\omega_{i_1} \dots \omega_{i_t})$ of length t; I^t a proper subset of Ω^t; $\Omega^{t-l} I^l$ the set of all possible sequences of events $(\omega_{i_1} \dots \omega_{i_{t-l}}, \omega_{i_{t-l+1}} \dots \omega_{i_t})$ such that $(\omega_{i_{t-l+1}} \dots \omega_{i_t}) \in I^l$, $l \leq t$; $p(I^l/\xi(t) = x)$ the probability that, from time $t+1$ to time $t+l$, the sequence $(\omega_{i_1} \dots \omega_{i_l}) \in I^l$ will appear, under the condition that $\xi(t) = x$; $p(\xi(t')/\xi(t'') = x)$ the probability distribution of the states of the chain $\xi(t)$ at time $t' \geq t''$, under the condition $\xi(t'') = x$.

Theorem 1. In the space of conditional measures on X, the weak distance*

$$L(p(\xi(t)/\xi(0) = x),\ p(\xi(t)/\xi(0) = y)) \leq \max(\alpha^l d,\ \tilde{\mu}_{t-l}),$$

where $l \leq t$,

$$\tilde{\mu}_{t-l} = \sup_{\Omega^{t-l} I^l} \left| p(\Omega^{t-l} I^l/\xi(0) = x) - p(\Omega^{t-l} I^l/\xi(0) = y) \right|.$$

Theorem 2. If

$$\sup_{\substack{\omega \in \Omega \\ t' \in [t, t+\tau]}} |p(\omega/\xi(t') = x) - p(\omega/\xi(t') = y)| \leq C_0 \rho(x, y),$$

then

$$|p(I^\tau/\xi(t) = x) - p(I^\tau/\xi(t) = y)| \leq C_0 \rho(x, y)/(1-a).$$

Let $I^1_{x_\omega}$ be the set of all those ω_i for which the mappings $\psi(x, \omega_i)$ have one and the same fixed point x_ω.

Theorem 3. Suppose:

1) For each time t there exists (generally speaking, for different t, different) set $I^1_{x_\omega}$ such that
$$p(I^1_{x_\omega}/\xi(t) = x) \geq \delta > 0;$$

2) the conditions of Theorem 2 are satisfied.

Then for every $l \leq t$,

$$\left| p(\Omega^{t-l} I^l/\xi(0) = x) - p(\Omega^{t-l} I^l/\xi(0) = y) \right| \leq \frac{a^m C_0 d}{1-a} + q_{m,t},$$

where $q_{m,t}$ is the probability of the absence of a run of m successes in a sequence of t Bernoulli trials with probability of success δ^2.**

From Theorem 3, in particular, follow conditions for the existence and uniqueness of a distribution μ, to which $p(\xi(t)/\xi(0) = x)$ converges weakly for all x. For the case when the BM has only two reactions, these conditions were established in ([4],[5]). The proof of Theorem 3 proceeds analogously to the corresponding proof in ([5]).

* The definition of weak distance see in ([6]), p. 147.

** It can be shown (see ([7]), pp. 326–327), that

$$q_{m,t} \sim \frac{1-\delta^2\lambda}{(m+1-m\lambda)(1-\delta^2)}\,\frac{1}{\lambda^{m+1}}, \quad \text{where } \lambda \approx 1+(1-\delta^2)\delta^{2m}+(m+1)(\delta^{2m}(1-\delta^2))^2.$$

To determine the degree of imitation of the BM by the approximating ABMk model, let us associate with the chain $\xi(t)$ the chain $\xi_k(t)$. The states of $\xi_k(t)$ are the fixed points of the mappings $\psi(\psi(\dots\psi(x,\omega_{i_1})\omega_{i_2})\dots\omega_{i_k})$. The chain $\xi_k(t)$ has a finite set of states

$$X_k = \{x_{\omega_{i_1}\dots\omega_{i_k}}\}$$

and passes from the state $x_{\omega_{i_1}\dots\omega_{i_k}}$ at time t to the state $x_{\omega_{i_2}\dots\omega_{i_k}\omega_j}$ with probability

$$p_t(\omega_j/\xi(t) = x_{\omega_{i_1}\dots\omega_{i_k}}).$$

Theorem 4. *Let $\xi(t)$ be the chain generated by the random behaviors of an MBM in a stationary environment, and let $\xi_k(t)$ be the corresponding chain generated by the random behaviors of an ABMk in the same stationary environment, with $k \geq 2$. Then:*

1) *If at least one of the chains $\xi_k(t)$, $\xi(t)$ has no absorbing sets, then $\xi_k(t)$ are ergodic chains, and $p(\xi(i)/\xi(0) = x)$ converges weakly to an invariant distribution μ independent of x, and the limiting distributions μ_k of the chains $\xi_k(t)$ converge weakly to μ as $k \to \infty$.*
2) *The chains $\xi_k(t)$ and $\xi(t)$ either both have, or both do not have, absorbing sets. In the former case the number of absorbing sets of $\xi_k(t)$ and $\xi(t)$ is the same, and the absorbing sets of $\xi_k(t)$ are proper subsets of the absorbing sets of $\xi(t)$.*
3) *The chains $\xi(t)$ and $\xi_k(t)$ either both have, or both do not have, absorbing states. These absorbing states are conditional fixed points $\|p_\omega(\beta/\alpha)\|$ (i.e., simple matrices $\|p_\omega(\beta/\alpha)\|$).*

Some cases in which a direct analytic computation of the limiting distributions of an MBM is possible are considered in (8).

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Note: Figure translations are in progress. See original paper for figures.