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Abstract

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ON SPACES OF CONSTRUCTIVE INFINITELY DIFFERENTIABLE FUNCTIONS AND ON FUNCTIONALS IN THEM

(Presented by Academician P. S. Novikov, 3 I 1968)

1. In the present note the terms and notation introduced in the papers $(^{1,2})$ are used. By a **word of type** we shall mean a complete code of an almost uniformly continuous function $(^{9})$. In $(^{9})$ an algorithm J was introduced with the following property: whatever the natural number n and the word $_1$ of type , $J(n\Box\, _1)$ is a primitive of order n of the complete code $_1$. Let $_1$ and $_2$ be words of type . If the condition

$$(\ _1,\ _2) \ \rightleftarrows\ \forall x(\ _2(x) -\ _2(0) = J(1\Box\ _1)(x))$$

is fulfilled, the function $_1$ is the derivative of the function $_2$. Then we shall also say that $_1$ is a **derivative** of $_2$. Introduce the formula

$$\mathfrak{P}_f \ \rightleftarrows\ (\langle\gamma\rangle \in (\mathrm{H} \to \ )) \ \&\ \forall n\,(\langle\gamma\rangle(n+1), \langle\gamma\rangle(n))$$

where γ is a variable for words in the alphabet $_0$. The elements of the set $\mathfrak{P}_f$ will be called **complete records of constructive infinitely differentiable functions**, or words of type f.

One can construct a formula $\mathfrak{E}_f$, algorithms $+_f$ of type $(ff \to f)$ and an algorithm $\dot{f}$ of type $(\mathbb{D}f \to f)$ such that, whatever the words f_1, f_2 of type f and the real duplex α,

$$\mathfrak{E}_f(f_1, f_2) \ \equiv\ (\langle f_1\rangle(0) \equiv \langle f_1\rangle(n)).$$

$$\forall n(\langle +_f(f_1\Box f_2)\rangle(n) \simeq \langle f_1\rangle(n) + \langle f_2\rangle(n));$$

$$\forall n(\langle \dot{f}(\alpha\Box f_1)\rangle(n) \simeq \alpha \cdot \langle f_1\rangle(n)).$$

By means of $\mathfrak{D}_f$ denote the record of an algorithm that transforms any natural number into the null element of type . By virtue of the results from (^{2}), § 11.2, one can construct an algorithm N_1 of type ($f \to$) such that, whatever the natural numbers m, n and the word f_1 of type f, $N_1(m\Box n\Box f_1)$ is an exact upper bound of the function $\langle f_1\rangle(n)$ on the segment $-m\Delta m$.

After this construct an algorithm N such that, for any m, n and f_1,

$$N(m\Box n\Box f_1) \simeq \max(N_1(m\Box 0\Box f_1)\Box \ldots \Box N_1(m\Box n\Box f_1)).$$

Let m be some fixed positive natural number. Introduce the formula

$$\mathfrak{P}_m \rightleftarrows \mathfrak{P}_f \ \& \ \forall x(|x| \geq m \supset \langle\gamma\rangle(0)(x) = 0).$$

It can be verified that the list*

$$_0,\ \mathfrak{P}_m,\ \mathfrak{E}_f,\ +_f,\ \dot{f},\ \mathfrak{D}_f,\ \widetilde{N}_m\Box \tag{1}$$

defines a constructive countably normed space (see the definition in (^{7})). This space will be denoted by K_m. It is pre-

is the space of complete records of all constructive infinitely differentiable functions that vanish outside the segment $-m\Delta m$. The points of the space K_m will be called **words of type** ϕ^m, the variables for which are denoted by $\phi_1^m, \phi_2^m, \ldots$. Being a computably normed space, K_m is a metrizable locally convex space (^{6,7}). The notions of completeness and separability introduced for metric spaces in (^{2,3}) are easily defined for the space K_m. The following holds.

Theorem 1. *The space K_m is complete and separable.*

Let $\mathfrak{A}$ and $\mathfrak{B}$ be sets of points of the space K_m. We shall say that $\mathfrak{A}$ is a **convex set** in K_m if

$$\forall \phi_1^m \phi_2^m xy\,(x + y = 1 \ \& \ \phi_1^m \in \mathfrak{A} \ \& \ \phi_2^m \in \mathfrak{A} \supset x \cdot \phi_1^m + y \cdot \phi_2^m \in \mathfrak{A}),$$

and that the set $\mathfrak{A}$ **absorbs** the set $\mathfrak{B}$ if

$$\exists x\,(x > 0 \ \& \ \forall \phi_1^m\,(\phi_1^m \in \mathfrak{B} \supset x \cdot \phi_1^m \in \mathfrak{A})).$$

Theorem 2. *In the space K_m one can construct a convex set $\mathfrak{A}$ possessing the properties: 1) $\mathfrak{A}$ absorbs every bounded set in the space K_m; 2) $\mathfrak{A}$ contains no neighborhood of zero in K_m.*

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* The meaning of the notation $N_m\Box$ is given in (^{3}).

2. Let $\mathfrak{M}$ be some constructive locally convex space and let λ be a linear operator mapping K_m into $\mathfrak{M}$. Let $\mathfrak{A}$ be a set of points of the space K_m. By $\lambda(\mathfrak{A})$ we denote the set of points of the space $\mathfrak{M}$ defined by the formula $\exists\phi_1^m(\phi_1^m \in \mathfrak{A} \& \lambda(\phi_1^m) = \theta_1)$, where θ_1 is a variable for points of $\mathfrak{M}$. The operator λ is called **bounded** if, for any bounded set $\mathfrak{A}$ in K_m, the set $\lambda(\mathfrak{A})$ is bounded in $\mathfrak{M}$; the operator λ is called **continuous** if, for any neighborhood $\mathfrak{B}$ of zero in $\mathfrak{M}$, there is a neighborhood $\mathfrak{A}$ of zero in K_m such that $\lambda(\mathfrak{A})$ is contained in $\mathfrak{B}$; the operator λ is called **p-continuous** if, for any sequence φ of points of the space K_m converging to zero, the sequence $(\lambda \circ \varphi)$ converges to zero in $\mathfrak{M}$. The notion of p-continuity was introduced for normed spaces in (^{5}).

Theorem 3. *There exist a constructive locally convex space $\mathfrak{M}$ and a linear operator λ, mapping K_m into $\mathfrak{M}$, such that λ is bounded and p-continuous, but not continuous.*

On the other hand, from the main theorem of (^{3}) and Theorem 1 it follows:

Theorem 4. *If $\mathfrak{M}$ is a metrizable locally convex space, then every linear operator mapping K_m into $\mathfrak{M}$ is continuous.*

3. Let us now consider functionals in the space K_m. By Theorem 4, every linear functional in K_m is continuous.

Denote by K'_m the locally convex space conjugate to the space K_m (see the definition of the conjugate space in (^{8})).

Recall that, according to the definition of the conjugate space introduced in (^{8}), a point of the space K'_m is a word of the form $P \odot Q$, where P is the record of a linear continuous functional in K_m, and Q is the record of its regulator of continuity. Let T $(T \doteq P \odot Q)$ be a point of the space K'_m. Then T denotes the functional whose record is the word P.

The variables for points of the space K'_m will be denoted by $F_1^m, F_2^m, \ldots$.

Let $\mathfrak{A}$ be a set of points of the space K_m and $\mathfrak{B}$ a set of points of the space K'_m. We shall say that the set $\mathfrak{B}$ is **bounded on the set** $\mathfrak{A}$ if there exists a positive real duplex a such that

$$\forall F_1^m \phi_1^m \, (F_1^m \in \mathfrak{B} \ \& \ \phi_1^m \in \mathfrak{A} \supset |F_1^m(\phi_1^m)| \leq a).$$

Theorem 5. There exists a bounded set $\mathfrak{B}$ of the space K'_m such that it is not bounded in any neighborhood of zero of the space K_m.

4. In the space K_m the locally convex topology corresponding to the multinorm $N_{m\square}$ will be called the **initial topology**. In addition to the initial topology, we define in K_m another topology in the following way.

Let $(A_r, \mathfrak{X}, \mathfrak{R})$ be a normal fundamental system of bounded sets in the space K'_m. Such a system can be constructed by the method indicated in (8). After this, construct a formula $\mathfrak{D}$ such that, whatever the element $\mathfrak{x}_1$ of the set $\mathfrak{X}$ and the point ϕ_1^m of the space K_m may be,

$$\mathfrak{D}(\mathfrak{x}_1, \phi_1^m) \equiv \forall F_1^m(\mathfrak{R}(\mathfrak{x}_1, F_1^m) \supset |F_1^m(\phi_1^m)| < 1).$$

It is not difficult to verify that the list $(A_r, \mathfrak{X}, \mathfrak{D})$ defines a locally convex topology on the linear space

$$\mathcal{C}h_0,\ \mathfrak{P}_m,\ \mathfrak{E}_f,\ +,\ \cdot,\ \mathfrak{D}_f.$$

We shall call this topology the **strong topology** in K_m. It can be shown that in K_m the strong topology majorizes the initial topology. But the converse is not true. Namely, we have:

Theorem 6. In the space K_m the initial topology does not majorize the strong topology.

Thus, in K_m the initial and strong topologies are not equivalent to one another. On the other hand, despite the nonequivalence of these topologies, it can be proved that in K_m every set bounded in the initial topology is also bounded in the strong topology, and conversely.

5. If in the list (1) m denotes a variable ranging over the positive natural numbers, then this list defines an expanding sequence of countably normed spaces, and hence an expanding sequence of locally convex spaces (8). It is easy to verify that in this sequence each space K_m is a β-closed (6) subspace of the space K_l $(l > m)$. By definition in (8), we can construct the strict inductive limit of the sequence of spaces K_m. This strict inductive limit is a locally convex space; we shall denote it by K^0. Let us note that the points of the space K^0 are words γ of type f satisfying the condition:

$$\exists m \forall x(|x| \geq m \supset \langle\gamma\rangle(0)(x) = 0).$$

In addition to the space K^0, for the sequence (1) we construct the so-called FR-strict inductive limit K as follows: as the points of the space K we take words of the form $m\tau\gamma$, where m is a positive natural number and γ is a word of type f satisfying the condition:

$$\forall x(|x| \geq m \supset \langle\gamma\rangle(0)(x) = 0).$$

The equality relation, linear operations and topology in K are defined in the analogous way as for the space K^0.

One may regard K as a constructive analogue of the space of all finite infinitely differentiable functions in classical mathematics (see (10)).

Theorem 7. There is no algorithm which transforms each point X of the space K^0 into a natural number m such that $m\tau X$ is a point of the space K.

Theorem 8. *The space K is multinormable.*

Let $\mathfrak{A}$ be a set of points of the space K. We shall say that $\mathfrak{A}$ is contained in K_m if from $n_t X \in \mathfrak{A}$ it follows that $X \in K_m$.

Theorem 9. *One can construct in the space K a bounded set $\mathfrak{A}$ such that there is no natural number m for which $\mathfrak{A}$ would be contained in K_m.*

This means that there exists a strict inductive limit of a sequence of locally convex spaces that is not regular (see ([8])).

Let us note that linear continuous functionals in the space K may be regarded as constructive analogues of generalized functions defined by Sobolev-Schwartz in classical mathematics.

Theorems 2, 5, 9 are proved with the aid of the universal arithmetical algorithm (see, for example, ([4])), and on the basis of these theorems Theorems 3, 6 are proved. Theorems 2-6, 9 show that certain assertions of classical functional analysis (see, for example, ([10], [11])) do not carry over to constructive mathematics.

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Note: Figure translations are in progress. See original paper for figures.

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