GENERALIZED MOTIONS IN RIEMANNIAN SPACES

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GENERALIZED MOTIONS IN RIEMANNIAN SPACES

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In the present note the concept of a group of generalized motions of an n-dimensional Riemannian space is introduced; it includes the definition of a group of motions. Generalized Killing equations are obtained, making it possible to clarify the geometric meaning of generalized motions. The question of the classification of solutions of the Einstein equations by groups of generalized motions is considered.

Let us consider a local Lie group H of point transformations of N-dimensional Euclidean space $E^N(x)$ into itself. A manifold $\mathfrak{M} \subset E^N(x)$ is called an invariant manifold of the group H if, for any point $x \in \mathfrak{M}$ and any transformation $T \in H$, one has $Tx \in \mathfrak{M}$. Denote by

$$X_\alpha = \xi^i_\alpha(x) \frac{\partial}{\partial x^i} \quad (\alpha = 1, \dots, r;\ i = 1, \dots, N)$$

the infinitesimal operators of the group H, and by R the common rank of the matrix $\|\xi^i_\alpha(x)\|$, where r is the order of the group H. The rank of an invariant manifold $\mathfrak{M}$ is the number $\rho = \dim \mathfrak{M} - R$. A manifold $\mathfrak{N} \subset E^N(x)$ is called a partially invariant manifold of the group H if $\mathfrak{N} \subset \mathfrak{M}$, where $\mathfrak{M}$ is some invariant manifold of this group not coinciding with $E^N(x)$. If as $\mathfrak{M}$ one takes the smallest invariant manifold containing $\mathfrak{N}$, then one can introduce the concepts of the rank and defect of invariance of a partially invariant manifold: the rank of the partially invariant manifold $\mathfrak{N}$ is the rank of $\mathfrak{M}$, and the defect of invariance is the number $(^1)$ $\delta = \dim \mathfrak{M} - \dim \mathfrak{N}$.

Let V_n be an n-dimensional Riemannian space with metric tensor given by the equations

$$g_{ij} = g_{ij}(x). \tag{1}$$

We shall regard transformations of the points of V_n as transformations of the points of the space $E^n(x)$; the Lie group of transformations will be denoted by H, and the infinitesimal operator will be written in the form

$$X = \xi^i(x)\frac{\partial}{\partial x^i} \quad (i = 1, \dots, n).$$

In addition, introduce the space $E^{n+m}(x, g)$, whose points are the coordinates x^i and the $m = n(n+1)/2$ components of the tensor g_{ij}. Then to the group H of transformations of the points of $E^n(x)$ there corresponds a group $\overline{H}$ of transformations of the points of $E^{n+m}(x, g)$, and the infinitesimal operator will have the form

$$\overline{X} = \xi^i(x)\frac{\partial}{\partial x^i} + \eta_{ij}(x, g)\frac{\partial}{\partial g_{ij}}.$$

Let us express the quantities $\eta_{ij}(x, g)$ in terms of $\xi^i(x)$. We proceed from the transformation law for the components of a tensor

$$g_{ij} = g'_{kl}\frac{\partial x'^k}{\partial x^i}\frac{\partial x'^l}{\partial x^j},$$

where the primes denote the transformed quantities (containing the transformation parameter t). Applying to both sides of this equality the operator

$$\left.\frac{\partial}{\partial t}\right|_{t=t_0}$$

(t_0 corresponds to the identity transformation), which commutes with the operator $\frac{\partial}{\partial x^i}$, we obtain

$$\eta_{ij}(x, g) = -g_{ki}\frac{\partial \xi^k}{\partial x^j} - g_{kj}\frac{\partial \xi^k}{\partial x^i}.$$

The equalities (1) define a certain manifold in the space $E^{n+m}(x, g)$.

Definition. The group $\bar{H}$ is called a **group of generalized motions of the Riemannian space** V_n if the manifold (1) is a partially invariant manifold of the group $\bar{H}$. The numbers ρ and δ will be called, respectively, the **rank** and the **defect** of the space V_n with respect to the group H.

Lemma. In order that the manifold

$$\mathfrak{M}: \quad \varphi^\nu(x, g) = 0 \qquad (\nu = 1, \dots, \mu;\ \mu \leqslant m) \tag{2}$$

be an invariant manifold of the group $\bar{H}$, it is necessary and sufficient that the generalized Killing equations

$$\left(\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ki}\frac{\partial \xi^k}{\partial x^j} + g_{kj}\frac{\partial \xi^k}{\partial x^i}\right) \frac{\partial \varphi^\nu}{\partial g_{ij}}\bigg|_{\mathfrak{M}} = 0 \qquad (\nu = 1, \dots, \mu). \tag{3}$$

Proof. From the condition $\mu \leqslant m$ and the invariance of the manifold $\mathfrak{M}$ it follows that

$$\left(\xi^k \frac{\partial \varphi^\nu}{\partial x^k} + \xi^k \frac{\partial \varphi^\nu}{\partial g_{ij}}\frac{\partial g_{ij}}{\partial x^k}\right)\bigg|_{\mathfrak{M}} = 0 \qquad (\nu = 1, \dots, \mu), \tag{4}$$

$$\bar{X}\varphi^\nu\big|_{\mathfrak{M}} \equiv \left[\xi^k \frac{\partial \varphi^\nu}{\partial x^k} - \left(g_{ki}\frac{\partial \xi^k}{\partial x^j} + g_{kj}\frac{\partial \xi^k}{\partial x^i}\right)\frac{\partial \varphi^\nu}{\partial g_{ij}}\right]\bigg|_{\mathfrak{M}} = 0 \qquad (\nu = 1, \dots, \mu). \tag{5}$$

Subtracting (5) from (4), we obtain (3). Conversely, from (3) and (4) follows (5), which is a necessary and sufficient condition for invariance of the manifold (2). The lemma is proved.

The smallest invariant manifold (assumed nonsingular) containing (1) can be given by the system of equations (1)

$$\Psi^\nu(I^1, \dots, I^t) \equiv \psi^\nu(x, g) = 0 \qquad (\nu = 1, \dots, \mu), \tag{6}$$

where $I^1(x,g), \dots, I^t(x,g)$ is a complete set of invariants of $\bar{H}$ and

$$t = n + m - R, \qquad \rho = \delta + n - R, \qquad \mu = m - \delta,$$

$$\max\{R - n, 0\} \leqslant \delta \leqslant \min\{R - 1, m - 1\}, \tag{7}$$

$$R = \operatorname{rank}\left\|\xi^i_\alpha(x),\, \eta^\alpha_{ij}(x,g)\right\|, \qquad \operatorname{rank}\left\|\frac{\partial \psi^\nu}{\partial g_{ij}}\right\| = \mu \qquad (\alpha = 1, \dots, r),$$

The manifold (6) will be called the **defining manifold of the space V_n** with metric tensor (1). Introduce the notation

$$h_{ij} = \xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ki}\frac{\partial \xi^k}{\partial x^j} + g_{kj}\frac{\partial \xi^k}{\partial x^i}.$$

The metric tensor of the space $\tilde{V}_n$, obtained from V_n by infinitesimal transformations of the group H, has the form (2)

$$\tilde{g}_{ij} = g_{ij} + h_{ij}\Delta t.$$

The following corollaries follow from the lemma.

Corollary 1. If H is a group of generalized motions of the space V_n with defining manifold (6), then for (1) the equations

$$h_i : \left.\frac{\partial \psi^\nu}{\partial g_i}\right|_{(1)} = 0 \qquad (\nu = 1, \dots, \mu).$$

are satisfied.

Corollary 2. In order that H be a group of motions of the space V_n, it is necessary and sufficient that the manifold (1) be invariant with respect to the group $\bar{H}$.

It follows from (8) that μ quantities h_{ij} are expressed linearly in terms of the remaining δ of them with variable coefficients. This means that the "distortion" of the space V_n occurs at the expense of an arbitrary change of these δ quantities h_{ij}. For $\delta = 0$ the group H becomes a group of motions and the "distortion" is absent. In other words, if under motions all components of

Table 1

No.	R	t	δ	ρ	μ	No.	R	t	δ	ρ	μ
1	1	13	0	3	10	21	7	7	5	2	5
2	2	12	0	2	10	22	7	7	6	3	4
3	2	12	1	3	9	23	8	6	4	0	6
4	3	11	0	1	10	24	8	6	5	1	5
5	3	11	1	2	9	25	8	6	6	2	4
6	3	11	2	3	8	26	8	6	7	3	3
7	4	10	0	0	10	27	9	5	5	0	5
8	4	10	1	1	9	28	9	5	6	1	4
9	4	10	2	2	8	29	9	5	7	2	3
10	4	10	3	3	7	30	9	5	8	3	2
11	5	9	1	0	9	31	10	4	6	0	4
12	5	9	2	1	8	32	10	4	7	1	3
13	5	9	3	2	7	33	10	4	8	2	2
14	5	9	4	3	6	34	10	4	9	3	1
15	6	8	2	0	8	35	11	3	7	0	3
16	6	8	3	1	7	36	11	3	8	1	2
17	6	8	4	2	6	37	11	3	9	2	1
18	6	8	5	3	5	38	12	2	8	0	2
19	7	7	3	0	7	39	12	2	9	1	1
20	7	7	4	1	6	40	13	1	9	0	1

the tensor g_{ij} are "carried along" without change (there are $t \geq m$ invariants $I^1(x,g), \dots, I^t(x,g)$ of the group $\bar{H}$, from which all g_{ij} can be found), then under generalized motions there are only $\mu \leq m$ invariants $\Psi^\nu(I^1, \dots, I^t)$, which make it possible to find μ components of the tensor g_{ij}, if the remaining δ components are prescribed.

Suppose we wish to determine whether there exists a subgroup H' of the group of generalized motions H that is a group of motions of the space V_n. For this it is enough to check whether δ quantities h_{ij} vanish for the coordinates $\xi^i(x)$ of the operators of the subgroup H'. If they vanish, then the others will be equal to zero by virtue of (8), i.e., the Killing equations $h_{ij} = 0$ will be satisfied.

As an example, let us consider the question of finding particular solutions of the Einstein equations in empty space

$$R_{ik} = 0 \qquad (i, k = 1, \dots, 4). \tag{9}$$

Calculations show that the broadest Lie group of transformations of the form

$$x'^i = f^i(x,g), \qquad g'_{ij} = F_{ij}(x,g),$$

admitted by equations (9), is limited to tensor transformations and one stretching $g'_{ij} = a g_{ij}$, $a = \text{const}$. If one considers only tensor transformations, then, according to the above-introduced definition, the classification of invariant or partially invariant solutions of (9) can be discussed as the classification of Einstein spaces by

groups of motions or generalized motions. In finding an Einstein space, the defect δ and rank ρ are involved: there are δ functions that cannot be found from the defining manifold (6), while the rank ρ indicates how many invariants in (6) will be independent variables. Therefore, the larger these two numbers are, the more difficult is the problem of finding the corresponding Einstein space. This approach makes it possible to speak of automodel solutions of the Einstein equations—solutions obtained on Abelian groups of motions [1]. For example, the cylindrical Einstein-Rosen waves constitute an automodel solution, while the Schwarzschild solution is non-automodel.

Table 1 gives the possible types of defining manifolds (6) of spaces V_4 admitting groups of generalized motions. This table was obtained by applying (7) for $n = 4$. Types 1, 2, 4, 7 of the table give groups of motions studied in [3].

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Note: Figure translations are in progress. See original paper for figures.

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