ON NECESSARY AND SUFFICIENT CONDITIONS UNDER WHICH A GIVEN SEQUENCE IS MINIMIZING

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 519.3:62-50

MATHEMATICS

E. S. LEVITIN

ON NECESSARY AND SUFFICIENT CONDITIONS UNDER WHICH A GIVEN SEQUENCE IS MINIMIZING

(Presented by Academician A. N. Tikhonov, 29 IV 1968)

Let E be a real Banach space. The problem of a conditional extremum, considered in general form, consists in minimizing a real functional $f_0(u)$ on some set $Q \subset E$, i.e., one seeks $\inf\limits_{u \in Q} f_0(u) = \mu$ and the points at which the lower bound is attained (if they exist). It is assumed that $\mu > -\infty$. In many extremal problems the set Q can be specified in the form $Q = \{u : f(u) \leq 0\} \cap S$, where $S \subset E$ is a closed set, and $f(u)$ is a real functional in E. In what follows it is assumed throughout that $f_0(u)$, $f(u)$ have a derivative at every point in every direction (^{1,2}) (and, consequently, are continuous at every point in every direction).

1°. Let the set $\mathfrak{M} = \{u^* \in Q : f_0(u^*) = \mu\}$ of minimum points of the functional $f_0(u)$ on Q be nonempty.

Theorem 1. Let S be a convex set. Then, for any $u^* \in \mathfrak{M}$, $\gamma(u^*) = 0$, where

$$\gamma(u^*) = \begin{cases} \inf\limits_{u \in S} \max\{f'(u^*, u - u^*),\ f_0'(u^*, u - u^*)\}, & \text{if } f(u^*) = 0, \\ \inf\limits_{u \in S} f_0'(u^*, u - u^*), & \text{if } f(u^*) < 0. \end{cases}$$

If $f_0(u)$ and $f(u)$ are convex functionals, and $\{u : f(u) < 0\} \cap S \neq \emptyset$, then the condition $\gamma(u^*) = 0$ is also sufficient for a minimum of $f_0(u)$ on Q at the point u^*.

Remark 1. Let $f_0(u)$ be a convex functional, and let $Q = S$. Then the condition $\gamma(u^*) = \inf\limits_{u \in S} f_0'(u^*, u - u^*) = 0$ is sufficient for a minimum at the point u^* even without the assumption of convexity of $S = Q$. Moreover, for any $v \in S$ the inequality

$$f_0(v) - \mu \leq \inf_{u \in S} f_0'(v, u - v)$$

is valid, giving an error estimate for finding the minimal value μ.

Remark 2. For the case when $f_0(u)$ is a Fréchet differentiable functional and $Q = S$, the necessary minimum condition $\gamma(u^*) = 0$, taking the form

$$\inf_{u\in S}\langle f_0'(u^*),\, u - u^*\rangle = 0,$$

was obtained in (^{3}). On its basis a numerical minimization method was also proposed there.

Remark 3. For the case when $E = L_{2,r}[0,T]$, $f_0(u) = \max\limits_{t\in[0,T]} R(t,u)$, where for every $t \in [0,T]$ the functional $R(t,u)$ is Fréchet differentiable, and $Q = S$ is a bounded convex set, the necessary minimum condition $\gamma(u^*) = 0$, taking the form

$$\inf_{u\in S} f_0'(u^*, u - u^*) = \inf_{u\in S}\max_{t\in A(u^*)}\int_0^T R_u'(t, u^*(\tau))(u(\tau) - u^*(\tau))\,d\tau = 0,$$

where

$$A(u^*) = \{t \in [0,T] : R(t,u^*) = f_0(u^*)\},$$

was obtained in (4). For the minimization in optimal-control problems of the differentiable functional $f_0(u)$ on the set S, in the presence of a constraint on the phase coordinates, a necessary minimum condition analogous to Theorem 1 was obtained in (5) (there, on its basis, a numerical minimization method was also considered).

Points $\tilde u \in Q$ satisfying the equation $\gamma(\tilde u) = 0$ will be called stationary points of $f_0(u)$ on Q. Let us note that if $\{u : f(u) < 0\} \cap S = \emptyset$, then every point $\tilde u \in Q$ will be stationary (indeed, in this case $f(u) \geqslant 0$ on S, and the equation $\gamma(u^*) = 0$ gives simply the minimum condition for $f(u)$ on S).

Suppose now that, for any fixed u, $f_0'(u,\bar u)$, $f'(u,\bar u)$ are linear convex functionals of $\bar u$ (1):

$$f_0'(u,\bar u) = \max_{l\in G_0(u)}\langle l,\bar u\rangle, \qquad f'(u,\bar u) = \max_{l\in G(u)}\langle l,\bar u\rangle,$$

where $G_0(u)$, $G(u)$ are weakly closed convex bounded sets in E', the space conjugate to E. Then

$$\max\{f_0'(u,\bar u), f'(u,\bar u)\} = \max_{l\in G_0(u)\cup G(u)}\langle l,\bar u\rangle.$$

Therefore the equation $\gamma(u^*) = 0$ takes the form

$$\gamma(u^*) = \inf_{u \in S} \max_{l \in H(u^*)} \langle l, u - u^* \rangle = 0,$$

where

$$H(u^*) = \begin{cases} G_0(u^*) \cup G(u^*), & \text{if } f(u^*) = 0, \\ G_0(u^*), & \text{if } f(u^*) < 0. \end{cases}$$

2°. For many extremal problems it is sufficient only to find, with a prescribed accuracy, the minimal value of the functional $f_0(u)$ on Q, and therefore for them it is sufficient to be able to construct minimizing sequences. On the other hand, a point of minimum may also fail to exist (for example, in a number of problems in the theory of optimal systems an optimal control does not exist, and there is only a so-called sliding regime). In this case the solution of the problem should be understood as a minimizing sequence (i.e., a sequence $u^n \in Q$ for which $f_0(u^n) \to \mu = \inf_{u \in Q} f_0(u)$).

Theorem 2. Let S be a closed convex set, and let $f_0(u)$, $f(u)$ be convex functionals, $\{u : f(u) < 0\} \cap S \neq \emptyset$.
Then every sequence $v^n \in Q$ for which

$$\lim_{n \to \infty} \gamma(v^n) = 0$$

is minimizing for $f_0(u)$ on Q (for any $v \in Q$, the functional $\gamma(v)$ is defined here in the same way as in Theorem 1).

Remark. If $Q = S$, then Theorem 2 is true even without the assumption on the convexity of S.

However, the condition

$$\lim_{n \to \infty} \gamma(v^n) = 0$$

may also fail to be necessary for the sequence $v^n \in Q$ to be minimizing. Let, for example, $E = R^1$ be the numerical axis, $f_0(u) = |u|$, $Q = S = [-1, +1]$. Then the sequence

$$v^n = (-1)^n \frac{1}{n}$$

is minimizing, but

$$\gamma(v^n) = \min_{|u| \leqslant 1} (\operatorname{sign} v^n)(u - v^n) \leqslant -1.$$

Such a situation is quite typical for nonsmooth $f_0(u)$, $f(u)$.

Theorem 3. Let S be a closed convex bounded set, the functional $f_0(u)$ uniformly continuous on the set $Q_\lambda = \{u : f_0(u) \leqslant \lambda\} \cap Q$ for some $\lambda > \mu$, and the functionals $f_0(u)$ and $f(u)$ differentiable in the sense of Fréchet and their gradients satisfy the Lipschitz condition on Q_λ. Further, suppose that either for all n, $f(v^n) = 0$, or for all n, $f(v^n) \leqslant -\delta < 0$. Under these conditions, in order that $v^n \in Q$ be a minimizing sequence, it is necessary that

$$\lim_{n\to\infty} \gamma(v^n) = 0.$$

Suppose now that S is an unbounded closed convex set. Let, for each $v \in S$, $S(v)$ be any closed convex set for which:

a) $v \in S(v) \subset S$;

b) $\forall u \in S,\ \sup\{\lambda : v + \lambda(u - v) \in S(v)\} \geq \rho_0 > 0$ for all $v \in S$;

c) $\sup\limits_{u\in S(v)} \|u - v\| \leq \rho$ for all $v \in S$.

Condition b) means that if $\bar{v}$ belongs to the cone of admissible directions (1) for the set S at its point v, then $v + \lambda\bar{v} \in S(v)$ for all $\lambda \in (0, \rho_0)$, and the number $\rho_0 > 0$ does not depend on $v \in S$. Condition c) means that $S(v)$ is a bounded set, and $\operatorname{diam} S(v) \leq 2\rho$ for all $v \in S$. For example, if S is a linear manifold, then as $S(v)$ one may take the intersection of S with the ball $\{u : \|u - v\| \leq r\}$. Conditions b), c) are then satisfied with $\rho_0 = \rho = r$.

Theorem 3 is also valid for the case of an unbounded S, if as $\gamma(v^n)$ one takes

$$\gamma(v^n) = \begin{cases} \inf\limits_{u\in S(v^n)} \max\left[\langle f_0'(v^n), u - v^n\rangle, \langle f'(v^n), u - v^n\rangle\right], & \text{if } f(v^n) = 0, \\ \inf\limits_{u\in S(v^n)} \langle f_0'(v^n), u - v^n\rangle, & \text{if } f(v^n) < 0. \end{cases}$$

3°. We now obtain a necessary condition for minimizing sequences for one, sufficiently general, class of nonsmooth functionals $f_0(u), f(u)$. Let $f_0(u) = J_0(u) + \varphi_0(u)$, where $\varphi_0(u) = \max\limits_{c_0\in A_0} \langle c_0, P_0(u)\rangle$; $f(u) = J(u) + \varphi(u)$, where $\varphi(u) = \max\limits_{c\in C} \langle c, P(u)\rangle$; here $J_0(u), J(u)$ are differentiable functionals; $P_0(u), P(u)$ are differentiable (in the Fréchet sense) operators from E into other B-spaces X_0, X, respectively; $A_0 \subset X_0'$, $A \subset X'$ are weakly closed bounded sets in the spaces conjugate to X_0, X. It can be shown that under these conditions, for any $v, \bar{v}$ there exist $f_0'(v, \bar{v}), f'(v, \bar{v})$, and moreover $f_0'(v, \bar{v}) = \langle J_0'(v), \bar{v}\rangle + \max_{c_0\in A_0(v)} \langle c_0, P_0'(v)\bar{v}\rangle$, where $J_0'(v)$ is the Fréchet gradient of the functional $J_0(v)$; $P_0'(v)$ is the gradient of the operator P_0 at the point v; $A_0(v) = \{c_0 \in A_0 : \langle c_0, P_0(v)\rangle = \varphi_0(v)\}$. Clearly, $f_0'(v, \bar{v}) = \max\limits_{l\in B_0(v)} \langle l, \bar{v}\rangle$, where $B_0(v) = \{l \in$

$E' : l = J_0'(v) = [P_0'(v)]^* c_0,\ c_0 \in A_0(v)\}$; here $[P_0'(v)]^*$ is the operator conjugate to $P_0'(v)$. Analogous formulas are also valid for $f'(v, \bar{v})$.

We give a condition under which the functional $\varphi_0(u) = \max\limits_{c_0 \in A_0} \langle c_0, P_0(u) \rangle$ is convex. Suppose that in the space X_0 a convex cone K_0 with vertex at zero is given (i.e., a convex set satisfying the condition $K_0 = \lambda K_0$ for any $\lambda > 0$), defining an order relation in X_0: $x \geq 0$ if and only if $x \in K_0$. Assume that the operator P_0 is convex (with respect to the cone K_0), i.e., for any $u_1, u_2 \in E$ and any $\alpha \in (0, 1)$

$$P_0(\alpha u_1 + (1 - \alpha) u_2) \leq \alpha P_0(u_1) + (1 - \alpha) P_0(u_2).$$

Further, let the set $A_0 \subset K_0^*$, where $K_0^* = \{x' \in X_0' : \langle x', x \rangle \geq 0,\ \forall x \in K_0\}$ is the cone conjugate to K_0. Then the functional $\varphi_0(u) = \max\limits_{c_0 \in A_0} \langle c_0, P_0(u) \rangle$ is convex.

Let $\delta > 0$. Denote by

$$A_0(v, \delta) = \{c_0 \in A_0 : \varphi_0(v) - \langle c_0, P_0(v) \rangle \leq \delta\},$$

and by

$$B_0(v, \delta) = \{l \in E' : l = J_0'(v) + [P_0'(v)]^* c_0,\ c_0 \in A_0(v, \delta)\}.$$

The sets $A(v, \delta), B(v, \delta)$ are defined analogously for the functional $f(v)$. Clearly, for any $\delta > 0$, $B_0(v) \subset B_0(v, \delta)$, $B(v) \subset B(v, \delta)$. Put

$$\gamma(v, \delta) = \inf_{u \in S_v} \max_{l \in H(v, \delta)} \langle l, u - v \rangle,$$

where $S_v = S$ if S is bounded, and $S_v = S(v)$ if S is an unbounded set;

$$H(v, \delta) = \begin{cases} B_0(v, \delta) \cup B(v, \delta), & \text{if } -\delta \leq f(v) \leq 0, \\ B_0(u, \delta), & \text{if } f(v) < -\delta. \end{cases}$$

Theorem 4. Let S be a closed convex set, the functional $J_0(u)$ and the operators $P_0(u), P(u)$ be uniformly continuous, and the gradients $J_0'(u)$,

$J'(u), P_0'(u), P'(u)$ satisfy the Lipschitz condition on $Q_\lambda = \{u : f_0(u) \leq \lambda\} \cap Q$ for some $\lambda > \mu$. In order that the sequence $v^n \in Q$ be minimizing for $f_0(u)$ on Q, it is necessary, and if $J_0(u), J(u), \varphi_0(u), \varphi(u)$ are convex functionals, $\{u : f(u) < 0\} \cap S \neq \emptyset$, then it is also sufficient, that

$$\lim_{\delta \to 0} \lim_{n \to \infty} \gamma(v^n, \delta) = 0.$$

In many extremal problems important for applications, the computation of the functional $\gamma(v, \delta)$, $v \in Q$, $\delta > 0$, is a problem of linear or quadratic programming.

On the basis of Theorem 4 one can propose a numerical method of minimization for nonsmooth extremal problems.

In conclusion, we note that in another form necessary and sufficient conditions for minimizing sequences (for convex problems) were obtained on the basis of duality theory by E. G. Gol' shtein (^{6}).

Moscow State University
named after M. V. Lomonosov

Received
15 III 1968

CITED LITERATURE

[1] A. Ya. Dubovitskii, A. A. Milyutin, *Zhurn. vychislit. matem. i matem. fiz.*, **5**, No. 3 (1965). [2] B. N. Pshenichnyi, *Kibernetika*, No. 5 (1965). [3] V. F. Dem' yanov, A. M. Rubinov, *Vestn. LGU*, issue 19 (1964). [4] V. F. Dem' yanov, *Vestn. LGU*, issue 2, No. 7 (1966). [5] V. F. Dem' yanov, *Vestn. LGU*, issue 3, No. 13 (1966). [6] E. G. Gol' shtein, *DAN*, **172**, No. 5 (1967).

Note: Figure translations are in progress. See original paper for figures.