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Applying the operation of taking the curl to the Navier–Stokes equations, we obtain equations in Helmholtz form, containing only projections of the velocity and their derivatives. As usual, we represent the true velocity in turbulent motion as the sum of the averaged components U_j and the fluctuating components u_j. Averaging the Helmholtz equations, subtracting these "averaged" equations from the original ones, and retaining only terms linear with respect to the fluctuating velocities and their derivatives, we find, for a uniform flow:

$$\frac{\partial^2 u_3}{\partial t\,\partial x_2}-\frac{\partial^2 u_2}{\partial t\,\partial x_3}+U_1\left(\frac{\partial^2 u_3}{\partial x_1\,\partial x_2}-\frac{\partial^2 u_2}{\partial x_1\,\partial x_3}\right)+\frac{\partial U_1}{\partial x_2}\frac{\partial u_3}{\partial x_1}=\nu\sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right),$$

$$\frac{\partial^2 u_1}{\partial t\,\partial x_3}-\frac{\partial^2 u_3}{\partial t\,\partial x_1}+U_1\left(\frac{\partial^2 u_1}{\partial x_1\,\partial x_3}-\frac{\partial^2 u_3}{\partial x_1^2}\right)+\frac{\partial U_1}{\partial x_2}\frac{\partial u_2}{\partial x_3}=\nu\sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right),\tag{1}$$

$$\frac{\partial^2 u_2}{\partial t\,\partial x_1}-\frac{\partial^2 u_1}{\partial t\,\partial x_2}+U_1\left(\frac{\partial^2 u_2}{\partial x_1^2}-\frac{\partial^2 u_1}{\partial x_1\,\partial x_2}\right)+\frac{\partial U_1}{\partial x_2}\frac{\partial u_3}{\partial x_3}-u_2\frac{\partial^2 U_1}{\partial x_2^2}=$$

$$=\nu\sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right).$$

In deriving system (1), the incompressibility condition for an incompressible fluid was used.

The direction ox_1 (denoted by ox) coincides with the direction of the flow; the axis ox_2 (denoted by oy) is directed upward from the flow axis. To simplify the notation, let us assume that the turbulence is plane, i.e. $u_3 \equiv 0$ and $\partial u_j/\partial x_3 \equiv 0$ (this assumption is not necessary). For a plane flow,

$$\frac{\partial^2 v}{\partial t\,\partial x} - \frac{\partial^2 u}{\partial t\,\partial y} + U\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x\,\partial y}\right) - v\frac{\partial^2 U}{\partial y^2} =$$

$$= \nu\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right],$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2}$$

System (2) is solved under the following boundary conditions. On the smooth solid boundary, the time-averaged shear stress at the wall $\tau_0 = \rho U_*^2$ and the space-time spectrum of the shear stresses at the wall $S_{\tau_0}(\omega, \chi)$ are assumed to be known.

For solving system (2), it is convenient to use an operator method. Applying to both equations (2) the two-dimensional Fourier transform with respect to time and to the coordinate x, we find

$$\frac{i\nu}{\chi}\frac{d^4 v^*}{dy^4} + \left(\frac{\omega}{\chi} + U - 2i\chi\nu\right)\frac{d^2 v^*}{dy^2} - \left(\omega\chi + U\chi^2 + \frac{\partial^2 U}{\partial y^2} - i\chi^3\nu\right)v^* = 0, \tag{3}$$

$$u^* = \frac{i}{\chi}\frac{dv^*}{dy}. \tag{4}$$

Here u^* and v^* are two-dimensional "images" of the velocity fluctuations "in Fourier space" ; ω and χ are the parameters of the Fourier transform in time and in the coordinate x, respectively

$$\left(u^* = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int u(x, y, t)\exp[-i(\omega t + \chi x)]\,dt\,dx\right).$$

In the viscous sublayer, the shear stresses in the mean velocity field substantially exceed the turbulent stresses. According to experimental data [(1)], this is the case at least for

$$y_+ = \frac{yU_*}{\nu} < 5.$$

Inside the viscous sublayer the mean velocities obey the linear law

$$U = U_* y_+. \tag{5}$$

Fig. 1. Distribution of the standards of longitudinal (u') and transverse (v') fluctuations in the viscous sublayer at a smooth wall. Experimental data: a—B. A. Fiedman [5,6], pressureless flow of water; b—J. Laufer [3], pressure flow of air; c—V. V. Orlov [4], pressureless flow of water

Figure 1: Fig. 1. Distribution of the standards of longitudinal (u') and transverse (v') fluctuations in the viscous sublayer at a smooth wall. Experimental data: a—B. A. Fiedman [5,6], pressureless flow of water; b—J. Laufer [3], pressure flow of air; c—V. V. Orlov [4], pressureless flow of water

Immediately at a smooth wall, equation (3) becomes

$$\left.\frac{d^3u^*}{dy^3}\right|_{y=0} = \frac{\tau_0^*}{\nu\rho}\left(2\chi^2 + \frac{i\omega}{\nu}\right), \tag{6}$$

where $\tau_0^* = \nu\rho\, du^*/dy$ is the image of the fluctuations of shear stress at the wall. Owing to the slow variation of the fluctuations inside the "viscous" sublayer, relation (6) may approximately be regarded as satisfied not only at the boundary, but also at a small distance from the wall inside the layer. In this case

$$d^5v^*/dy^5\big|_{y=0} = 0$$

and, differentiating (3), we find as $y \to 0$

$$d^3v^*/dy^3\big|_{y=0} = B_1\tau_0^* + iB_2\tau_0^*, \tag{7}$$

where

$$B_1 = -\frac{2}{\omega^2/\chi^2 + 4}, \qquad B_2 = \frac{\omega}{\omega^2/\chi^2 + 4\chi\nu}. \tag{7a}$$

Linear dimensions, wave numbers χ, and frequencies ω here and below are measured in units that are combinations of the parameters ν, ρ, U_*,

$$[L] = \nu/U_*, \qquad [t] = \nu/U_*^2, \qquad [\tau_0] = \rho U_*^2.$$

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Representing the images of the fluctuations in the form of power series in y, multiplying the complex conjugate values of the images and calculating the mathematical expectation, we find the spectra of the components of velocity fluctuations and of turbulent stress:

$$S_u(\omega,\chi,y)=\left\{y^2-\frac{B_2}{\chi}y^3+\left(\frac{B_1^2+B_2^2}{4}-8\frac{\omega}{\chi}\right)y^4+...\right\}S_\tau(\omega,\chi), \tag{8}$$

$$S_v(\omega,\chi,y)=\left\{\frac{\chi^2}{4}y^4-\frac{\chi}{6}B_2y^5+\left(\frac{B_1^2+B_2^2}{36}-\chi\omega\right)y^6+...\right\}S_\tau(\omega,\chi), \tag{9}$$

$$S_{uv}(\omega,\chi,y)=\left\{\frac{i\chi}{2}y^3-\frac{B_1-i5B_2}{12}y^4+\left[-2\chi^2+i\left(\frac{B_1^2+B_2^2}{12\chi}-3\omega\right)\right]y^5+...\right\}\times$$

$$\times S_\tau(\omega,\chi). \tag{10}$$

Integrating expressions (8)–(10) over the limits from $-\infty$ to ∞, we can find the relative values of the standards of the fluctuations and the turbulent friction:

$$u'=\sqrt{\langle u^2\rangle}\approx a_1y, \tag{11}$$

$$v'=\sqrt{\langle v^2\rangle}\approx a_2y^2, \tag{12}$$

$$\langle uv\rangle\approx -a_3y^4. \tag{13}$$

The proportionality coefficients have the following physical meaning:

$$a_1=\left[\iint_{-\infty}^{\infty}S_{\tau_0}(\omega,x)\,d\omega\,dx\right]^{1/2}=\tau_0', \tag{14}$$

the standard deviation of fluctuations of the shear stress on the wall (according to experimental data $a_1=0.3$, see Fig. 1);

$$a_2=\frac{1}{2}\left[\iint_{-\infty}^{\infty}x^2S_{\tau_0}(\omega,x)\,d\omega\,dx\right]^{1/2}=\frac{1}{2}\left(\frac{\partial\tau_0}{\partial x}\right)', \tag{15}$$

one half of the standard deviation of the gradients of the fluctuations of the shear stress on the wall ($a_2=0.020\div 0.005$, see Fig. 2);

$$a_3=\frac{1}{8}\left\langle(\partial^2\tau_0/\partial x^2)^2\right\rangle-\frac{1}{24}\left\langle(\partial\tau_0/\partial t)^2\right\rangle. \tag{16}$$

The form of the spectrum of the shear stress at the boundary $S_{\tau_0}(\omega, x)$ can be very roughly estimated from the measurement data of Klebanoff and Laufer (Fig. 3), assuming that up to values $y = 3 \div 6$ the representation

$$S_u(\omega, x, y) \approx \varphi(y) S_{\tau_0}(\omega, x),$$

valid for small y, is permissible.

Fig. 2. Distribution of the standard deviations of transverse fluctuations in the viscous sublayer. The notation is the same as in Fig. 1

Fig. 3. Normalized spectra of longitudinal velocity fluctuations: 1 —according to measurements by J. Laufer (3), $y_+ = 3.05$; 2 —according to measurements by P. Klebanoff (2), $y_+ = 5.95$; 3 —assumed normalized spectrum of the shear stress on the wall

To pass from frequency spectra (Fig. 3) to longitudinal spectra, which are necessary for calculating the coefficients a_2 and a_3, we shall use the hypothesis of "frozen turbulence," transported with velocity $U_k \sim y$ (in fractions of U_*). Integrating the spectrum shown in Fig. 3, we find

$$\frac{2a_2}{a_1} \approx \frac{1}{U_k} \left[\int_0^\infty \omega^2 S_{\tau_0}(\omega)\, d\omega \Big/ \int_0^\infty S_{\tau_0}(\omega)\, d\omega \right]^{1/2} = 0.03;$$

this ratio corresponds to estimates obtained by another method from the data of Figs. 1 and 2,

$$2a_2/a_1 = 0.10 \div 0.03.$$

Using Fig. 3, we also find

$$\left\langle \left(\frac{\partial^2 \tau_0}{\partial x^2} \right)^2 \right\rangle \approx \frac{2}{U_k} \int_0^\infty \omega^4 S_{\tau_0}(\omega)\, d\omega = 3.2 \cdot 10^{-4} (\tau_0')^2,$$

$$\left\langle \left(\frac{\partial \tau_0}{\partial t} \right)^2 \right\rangle = 2 \int_0^\infty \omega^2 S_{\tau_0}(\omega)\, d\omega = 0.8 \cdot 10^{-2} (\tau_0')^2.$$

Hence, from (16) we obtain $a_3 = -2.9 \cdot 10^{-4} (\tau_0')^2 < 0$.

Thus, immediately at the wall the turbulent stresses have the opposite sign. This may mean, in particular, that in the viscous sublayer the mean flow receives energy from turbulent fluctuations, and not the other way around, as is usually the case in turbulent flows. This conclusion does not contradict the results of careful measurements performed by B. A. Fidman ([6]).

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CITED LITERATURE

1. A. S. Monin, A. M. Yaglom, *Statistical Hydromechanics*, Part I, Moscow, 1965.

2. P. S. Klebanoff, "Characteristics of Turbulence in a Boundary Layer with Zero Pressure Gradient," NACA, Rep. 1247, 1955; TN 3178, 1954.

3. J. Laufer, *J. Aeronaut. Sci.*, May, 277 (1950).

4. V. V. Orlov, *Prikl. mekh. i tekhn. fiz.*, No. 4 (1966).

5. B. A. Fidman, in: *Problems of Channel Processes*, Leningrad, 1953.

6. B. A. Fidman, in: *Problems of Hydraulic Engineering Structures*, Publishing House of the Academy of Sciences of the USSR, 1960.

Note: Figure translations are in progress. See original paper for figures.