ON A BOUNDARY-VALUE PROBLEM FOR THE SYSTEM OF NAVIER-STOKES EQUATIONS

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Abstract

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ON A BOUNDARY-VALUE PROBLEM FOR THE SYSTEM OF NAVIER-STOKES EQUATIONS

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Let Q be a domain in three-dimensional space $x = (x_1, x_2, x_3)$, bounded above by the plane $\gamma_0 : x_3 = 0$, and below by the surface $\gamma_1 : x_3 = \varphi(x_1, x_2)$; $-H \leqslant \varphi(x_1, x_2) \leqslant 0$, where H is a certain positive constant. The domain Q may be either bounded in the directions perpendicular to the axis Ox_4, or unbounded. The line of intersection of γ_0 and γ_1, if it exists, will be denoted by S. Consider in the domain Q the stationary linearized system of Navier-Stokes equations

$$\nu \Delta \mathbf{v} + \mathbf{F} = \operatorname{grad} p, \qquad \operatorname{div} \mathbf{v} = 0, \tag{1}$$

where the parameter $\nu > 0$, and $\mathbf{F}(x) = (F_1(x), F_2(x), F_3(x)) \in \mathcal{L}_2(Q)$ is a given vector. We are interested in the question of unique solvability of the system (1) under the boundary conditions

$$\mathbf{v}|_{\gamma_1} = 0, \tag{2}$$

$$v_{1x_3} + v_{3x_1}|_{\gamma_0} = v_{2x_3} + v_{3x_2}|_{\gamma_0} = p - 2\nu v_{3x_3}|_{\gamma_0} = 0, \tag{3}$$

where $\mathbf{v}(x) = (v_1(x), v_2(x), v_3(x))$ is the unknown velocity vector, and $p(x)$ the unknown pressure.

The corresponding nonstationary problem is posed analogously. In the cylinder $\Omega_T = Q \times (0, T)$ one is required to find a vector $\mathbf{v}(x, t)$ and a function $p(x, t)$ satisfying in Ω_T the nonstationary linearized Navier-Stokes system

$$\mathbf{v}_t + \operatorname{grad} p = \nu \Delta \mathbf{v} + \mathbf{F}(x, t), \qquad \operatorname{div} \mathbf{v} = 0, \tag{4}$$

satisfying on the boundary the conditions

$$v|_{\Gamma_1} = 0, \qquad v_{1x_3} + v_{3x_1}|_{\Gamma_0} = v_{2x_3} + v_{3x_2}|_{\Gamma_0} = p - 2\nu v_{3x_3}|_{\Gamma_0} = 0, \tag{5}$$

where $\Gamma_1 = \gamma_1 \times (0, T)$, $\Gamma_0 = \gamma_0 \times (0, T)$, $\mathbf{F} \in \mathcal{L}_2(\Omega_T)$, and the initial condition at $t = 0$

$$\mathbf{v}|_{t=0} = 0. \tag{6}$$

Physically, problem (1)–(3) (as well as the corresponding nonstationary problem (4)–(6)) is the problem of finding the velocity $\mathbf{v}$ of the particles of a viscous incompressible fluid and the pressure p, when on the bottom γ_1 the no-slip condition is prescribed, while on the free surface γ_0 the conditions (3) of vanishing of the three components of the stress tensor are prescribed ([1]). Together with problems (1)–(3) and (4)–(6), one may also consider more complicated problems, for example problems with rotation. By this is meant the problem of finding, in the domain Q (in the stationary case), a solution of the system

$$\nu \Delta \mathbf{v} + [\vec{\omega} \times \mathbf{v}] + \mathbf{F}(x) = \operatorname{grad} p, \qquad \operatorname{div} \mathbf{v} = 0, \tag{1'}$$

satisfying conditions (2)–(3), where $\vec{\omega}$ is a certain bounded vector measurable in Q. The case when there is no free surface was studied in important papers ([2–4, 7])*.

* *Note added in proof.* Recently the author became acquainted with paper [8], in which related questions are considered.

1. Denote by $H_1(Q)$ the Hilbert space of vector-functions $\{\mathbf{u}(x)\}$ obtained by completing, with respect to the norm

$$\|\mathbf{u}\|_{H_1(Q)} = \left(\iiint_Q \sum_i |\operatorname{grad} u_i|^2 \, dx \right)^{1/2} \tag{7}$$

the set of solenoidal vectors $\mathbf{u}(x)$ that vanish in a boundary strip near the portion of the boundary γ_1.

Lemma 1. In the Hilbert space $H_1(Q)$, instead of the norm defined by (7), one may introduce the equivalent norm

$$\|\mathbf{u}\|'_{H_1(Q)} = \left(\iiint_Q \sum_{ij} (u_{ix_j} + u_{jx_i})^2 \, dx \right)^{1/2}.$$

We define a generalized solution of problem (1)–(2)–(3) as follows. A vector $\mathbf{v}$ and a function p are called a generalized solution of problem (1)–(2)–(3) if $\mathbf{v} \in H_1(Q)$, $p \in \mathcal{L}_2(Q)$, and for every vector-function $\mathbf{u} \in H_1(Q)$ the equality

$$\frac{\nu}{2} \iiint_Q \sum_{ij} (u_{ix_j} + u_{jx_i})(v_{ix_j} + v_{jx_i}) \, dx = \iiint_Q \sum_i F_i u_i \, dx \tag{8}$$

holds.

The correctness of this definition follows from Lemma 2.

Lemma 2. If $\mathbf{v}, p$ is an almost everywhere solution of problem (1)–(2)–(3) (and, a fortiori, a classical solution of this problem), then equality (8) holds for every $\mathbf{u} \in H_1(Q)$. If $\mathbf{v} \in H_1(Q) \cap \overset{\circ}{W}{}_2^2(Q)$ and (8) holds for every $\mathbf{u} \in H_1(Q)$, then there exists a function $p \in W_2^1(Q)$ such that $(\mathbf{v}, p)$ is an almost everywhere solution of problem (1)–(2)–(3). Moreover, if V_δ is an arbitrary δ-neighborhood, $\delta > 0$, of the curve S, and Q' is any compact part of the closure $\overline{Q}$ of the domain Q, and if $\mathbf{v} \in H_1(Q) \cap W_2^2(Q' \setminus V_\delta)$, then there exists a function $p \in W_2^1(Q' \setminus V_\delta)$ such that in $Q' \setminus V_\delta$ $(\mathbf{v}, p)$ is an almost everywhere solution of problem (1)–(2)–(3).

Using the Riesz representation theorem for a bounded linear functional in a Hilbert space and Lemma 1, we obtain Theorem 1.

Theorem 1. For every vector-function $\mathbf{F} \in \mathcal{L}_2(Q)$ there exists a unique generalized solution of problem (1)–(2)–(3).

2. The nonstationary case is considered analogously.

Let $H_2(\Omega_T)$ be the Hilbert space of vector-functions obtained after completing the set of solenoidal vectors $\{\mathbf{v}(x,t)\}$ satisfying the boundary conditions (5) and the initial conditions (6), with respect to the norm of the space $W_2^1(\Omega_T)$:

$$\|\mathbf{v}\|_{H_2(\Omega_T)} = \left(\int_0^T dt \iiint_Q \left(\sum_i |\operatorname{grad} v_i|^2 + |v_{it}|^2 \right) dx \right)^{1/2}.$$

A vector $\mathbf{v}(x,t) \in H_2(\Omega_T)$ and a function $p(x,t) \in \mathcal{L}_2(\Omega_T)$ will be called a generalized solution of problem (4)–(5)–(6) if the equality

$$\int_0^T dt \iiint_Q (u_t, v_t) e^{-t}\, dx + \frac{\nu}{2} \int_0^T dt \iiint_Q \sum_{ij} (v_{ix_i} + v_{ix_j})(u_{itx_j} + u_{jtx_i}) e^{-t}\, dx =$$

$$= \int_0^T dt \iiint_Q \sum_i F_i u_{it} e^{-t}\, dx \tag{9}$$

holds for every vector-function $\mathbf{u}(x,t) \in H_3(\Omega_T)$, where $H_3(\Omega_T)$ is the Hilbert space of vector-functions belonging to $H_2(\Omega_T)$ and such that the vector $\mathbf{u}_t$ has a square-integrable gradient in the domain Ω_T.

The uniqueness of such a generalized solution follows directly from Lemma 1: for $\mathbf{F} \equiv 0$ we take

$$\mathbf{u} = \int_0^t e^t \mathbf{v}(x,t)\, dt$$

and substitute it into (9) as a test function. To prove the existence of a generalized solution, consider the left-hand side of equality (9) as a linear bounded functional on $\mathbf{v} \in H_2(\Omega_T)$ for any fixed element $\mathbf{u} \in H_3(\Omega_T)$. Then, according to the Riesz theorem, (9) can be rewritten in the form

$$[\mathbf{v}, A\mathbf{u}] = (\mathbf{F}, \mathbf{u}), \tag{10}$$

where the parentheses (,) denote the scalar product in $\mathcal{L}_2(\Omega_T)$ with weight e^{-t}, and [,] denotes the scalar product in $H_2(\Omega_T)$ with the same weight; the operator A is, generally speaking, an unbounded operator from $H_3(\Omega_T)$ into $H_2(\Omega_T)$. However, it can be shown that the operator A has a bounded inverse; hence equation (10), rewritten in the form $[A^*\mathbf{v}, \mathbf{u}] = [\Phi, \mathbf{u}]$, where $[\Phi, \mathbf{u}] = (\mathbf{F}, \mathbf{u})$ is the realization of the functional $(\mathbf{F}, \mathbf{u})$ in the metric of $H_2(\Omega_T)$, has a solution $\mathbf{v} = A^{*-1}\Phi \in H_2(\Omega_T)$. Thus, the following holds.

Theorem 2. *For any $\mathbf{F} \in \mathcal{L}_2(\Omega_T)$, problem (4)–(5)–(6) has a unique generalized solution from $H_2(\Omega_T)$.*

3. In the case of problem $(1')$–(2)–(3), the generalized solution is defined as a solution in the space $H_1(Q)$ of the operator equation $\mathbf{v} + A\mathbf{v} = \Phi$, where the operator A is defined by the equality

   $$\{A\mathbf{v}, \mathbf{u}\} = \int_Q (\vec{\omega}\, \mathbf{v}\mathbf{u})\, dx$$

   (where $(\vec{\omega}\, \mathbf{v}\mathbf{u})$ denotes the mixed product $([\omega \times \mathbf{v}]\mathbf{u})$), and

   $$\{\Phi, \mathbf{u}\} = \int_Q (\mathbf{F}, \mathbf{u})\, dx.$$

   The operator A is completely continuous and skew-symmetric (since

   $$(\vec{\omega}\, \mathbf{v}\mathbf{u}) = -(\vec{\omega}\, \mathbf{u}\mathbf{v})$$

   ); therefore Theorem 1 remains valid also in this case.

4. The solutions of the stationary and nonstationary problems constructed in the preceding sections in fact possess greater smoothness than that established in Theorems 1 and 2.

Theorem 3. *If $\mathbf{F}(x) \in W_p^l(Q)$, $p > 1$, and l is a nonnegative integer, then the generalized solution from $H_1(Q)$ of problem (1)–(2)–(3) has the following properties: $\mathbf{v} \in W_p^{l+2}(Q')$, $p(x) \in W_p^{(l+1)}(Q')$, where Q' is any part of Q such that $\overline{Q'} \subset Q$. Further, if δ is an arbitrary positive number and the function $\varphi(x_1, x_2)$ occurring in the definition of the surface γ_1 is sufficiently smooth, then $\mathbf{v} \in W_p^{(l+2)}(Q'' \setminus V_\delta)$, $p(x) \in W_p^{(l+1)}(Q'' \setminus V_\delta)$, where Q'' is any compact part of $\overline{Q}$, and V_δ is the δ-neighborhood of the line S where the boundaries γ_0 and γ_1 meet.*

Theorems 3 and 2 also make it possible to establish the corresponding result for the nonstationary problem.

Theorem 4. *If* $\mathbf{F}(x,t) \in W^{2l,l}_{p,x,t}(\Omega_T)$, $p > 1$, *and* l *is a nonnegative integer, then the generalized solution from* $H_2(\Omega_T)$ *of problem (4)–(5)–(6) has the following properties:* $\mathbf{v}(x,t) \in W^{2l+2,l+1}_{p,x,t}(\Omega'_T)$, $p(x) \in W^{2l+1,l}_{p,x,t}(\Omega'_T)$, *where* Ω'_T *is any part of* Ω_T *such that* $\overline{\Omega'_T} \subset \Omega_T$. *On the other hand, if* $\delta > 0$ *is arbitrarily small and the function* $\varphi(x_1,x_2)$ *is sufficiently smooth, then* $\mathbf{v}(x,t) \in W^{2l+2,2l+1}_{p,x,t}(\Omega''_T \setminus (V_\delta \times (0,T)))$, $p(x,t) \in W^{2l+1,l}_{p,x,t}(\Omega''_T \setminus (V_\delta \times (0,T)))$, *where* Ω''_T *is any compact part of* $\overline{\Omega_T}$.

In the proof of Theorem 3 we shall confine ourselves to studying the smoothness of the generalized solution near the portion $\gamma_0 \setminus S$ of the boundary, since the required interior smoothness, as well as the smoothness of the solution near the portion $\gamma_1 \setminus S$ of the boundary, can be established in the same way as in the book (4). We take

$$\mathbf{u}(x) = \mathbf{u}^{(1)}_N(x) - \mathbf{u}^{(2)}_N(x), \qquad q(x) = q^1_N(x,y) - q^2_N(x,y), \tag{11}$$

where

$$\mathbf{u}^{(1)}_N = \frac{1}{\nu}\,\mathrm{rot}(\zeta_\delta(x)(\mathrm{rot}\,\vec{\Phi}_{N,k}(x-y)));$$

y is a parametric point; δ is a sufficiently small positive number; $\zeta_\delta(x)$ is a smooth function equal to zero in the $\delta/2$-neighborhood of γ_1 and equal to 1 outside the δ-neighborhood of γ_1,

$$\vec{\Phi}_{N,k}(x) = -\frac{e^k}{(2\pi)^3}\left\{\int_{|\alpha'|\geqslant 1} d\alpha' \int_{-\infty}^{+\infty} \frac{\theta_N(\alpha)e^{i(\alpha,x)}}{|\alpha|^4}\,d\alpha_3 + \int_{|\alpha'|<1} d\alpha' \int_l \frac{\theta_N(\alpha)e^{i(\alpha,x)}}{|\alpha|^4}\,d\alpha_3\right\},$$

where $\theta_N(\alpha)$ is a certain function analytic in $\mathrm{Im}\,\alpha_3 > 0$, sufficiently rapidly decreasing on the real plane, and, as $N \to \infty$, uniformly on every compact set tending to 1 (^{5}); l is a contour in the α_3-plane consisting of pieces of the real axis $(-\infty,-1)$ and $(1,+\infty)$ and of the semicircle $|\alpha_3| = 1$, $\mathrm{Im}\,\alpha_3 \geqslant 0$; $\alpha' = (\alpha_1,\alpha_2)$, $\alpha = (\alpha',\alpha_3)$; e^k is the unit vector directed along the axis OX_k. As the function $q^1_N(x,y)$ we take the function $\nu\,\mathrm{div}(\zeta_\delta(x)\Delta\vec{\Phi}_{N,k})$. The vector-function $\mathbf{u}^2_N$ is defined as the vector $\mathrm{rot}(\zeta_\delta(x)\vec{\theta}(x,y))$, where the vector $\vec{\theta}$ in the half-space $x_3 < 0$ is specified by its Fourier transform with respect to x_1 and x_2 as follows:

$$\tilde{\theta}_1(\alpha_1,\alpha_2,x_3;y) = \left[\tilde{\sigma}_1\left(\frac{\alpha_2^2}{|\alpha|^4} + \frac{\alpha_1^2}{2|\alpha|^3}x_3\right) - \tilde{\sigma}_2\frac{\alpha_1\alpha_2}{2|\alpha|^3}\left(\frac{2}{|\alpha|} - x_3\right) + \tilde{\sigma}_3\frac{i\alpha_1\alpha_2}{2\nu|\alpha|}\left(\frac{x_3}{|\alpha|} - \frac{1}{|\alpha|^2}\right)\right]e^{|\alpha|x_3},$$

$$\tilde{\theta}_2(\alpha_1, \alpha_2, x_3; y) = \left[-\tilde{\sigma}_1 \frac{\alpha_1 \alpha_2}{2|\alpha|^3} \left(\frac{2}{|\alpha|} - x_3\right) + \tilde{\sigma}_2 \left(\frac{\alpha_1^2}{|\alpha|^4} + \frac{\alpha_2^2}{2|\alpha|^3} x_3\right) + \tilde{\sigma}_3 \frac{i\alpha_2}{2\nu|\alpha|} \left(\frac{x_3}{|\alpha|} - \frac{1}{|\alpha|^2}\right)\right] e^{|\alpha| x_3}, \qquad \tilde{\theta}_3 \equiv 0,$$

and $\tilde{\sigma}_1 = \tilde{\sigma}_1(\alpha_1, \alpha_2; y)$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$ are the Fourier transforms with respect to x_1, x_2, at $x_3 = 0$, of the functions respectively

$$u^{(1)}_{N1x_3} + u^1_{N3x_1}, \quad u^1_{N2x_3} + u^1_{N3x_2}, \quad q^{(1)}_N - 2\nu u^1_{N3x_3}.$$

The Fourier transform of the function $q^2_N(x, y)$ in the half-space $x_3 < 0$ is defined by

$$\tilde{q}^2_N = e^{|\alpha| x_3} \left(\tilde{\sigma}_3 - \nu \left(\frac{\alpha_1}{|\alpha|}\tilde{\sigma}_1 + \frac{\alpha_2}{|\alpha|}\tilde{\sigma}_2\right)\right).$$

The functions (11) may be substituted into (8) as test functions. In doing so, transferring all derivatives to the function **u** by integration by parts, we obtain

$$v_{k,N}(y) = \iiint\limits_Q \sum_i F_i(x)(u^1_{iN} - u^2_{Ni})\, dx + B_N(y, \mathbf{v}), \tag{12}$$

where

$$\lim_{N\to\infty} \|v_{k,N}(y) - v_k(y)\|_{L_p(Q_\delta)} = 0,$$

and $B_N(y, \mathbf{v})$ is a linear functional of $\mathbf{v} \in W^1_p(Q)$, uniformly bounded in N, such that the functional $D^2_y B(y, \mathbf{v})$ is also uniformly bounded in N in $W^1_p(Q)$, where D^2_y is an arbitrary second derivative with respect to y. From equality (12), with the help of the theorems on multipliers (^{6}), the required result is obtained.

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Note: Figure translations are in progress. See original paper for figures.

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