ON THE GROWTH OF PLURISUBHARMONIC FUNCTIONS

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Abstract

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MATHEMATICS

L. I. RONKIN

ON THE GROWTH OF PLURISUBHARMONIC FUNCTIONS

AND ON THE DISTRIBUTION OF VALUES OF ENTIRE FUNCTIONS

OF SEVERAL VARIABLES

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In the works of M. Sire (1), P. Lelong (2), and the author (3) it was shown that, in a certain sense, the growth of an entire function $f(z_1, \ldots, z_n, w)$ with respect to the variable w is the same for almost all $z_1, \ldots, z_n$. It is natural to suggest that an analogous situation also holds for the function $n_f(t; z_1, \ldots, z_n)$, equal to the number of zeros of the function $f(z_1, \ldots, z_n, w)$ in the disk $|w| \le t$. Let us note that the methods by which, in (1−3), the growth of the function $f(z_1, \ldots, z_n, w)$ was investigated are not suitable for the investigation of the function $n_f(t; z_1, \ldots, z_n)$, since they are based on the expansion of the function $f(z_1, \ldots, z_n, w)$ in a power series in w. In the special case when

$$f(z_1, w) = \varphi(z_1) - w,$$

the behavior of the function $n_f(t; z_1)$ was studied in the theory of the distribution of values of entire and meromorphic functions of one variable. In this note we introduce and investigate a certain special class of plurisubharmonic functions. From the results obtained thereby there follow, in particular, both some of the theorems contained in (1−3) on the growth of an entire function and the corresponding assertions on the growth of the function $n_f(t; z_1, \ldots, z_n)$.

Let us introduce the following notation and definitions.

We denote by C^n the space of complex variables $z_1, \ldots, z_n$. By $C(E)$, where $E \subset C^1$, we denote the inner capacity of the set E.

By $\Delta(E; z'_1, \ldots, z'_{n-1})$, where the set $E \subset C^n$, we denote the intersection of E with the plane $\{(z_1, \ldots, z_n); z_1 = z'_1, \ldots, z_{n-1} = z'_{n-1}\}$.

For $n > 1$, by $\Gamma^1_{n-1}(E)$, where the set $E \subset C^n$, we denote the set of those points $(z_1, \ldots, z_{n-1})$ for which $C(\Delta(E; z_1, \ldots, z_{n-1})) > 0$. We also put $\Gamma^1_1(E) = E$ and $\Gamma^1_n(E) = \Gamma^1_2(\Gamma^2_3(\ldots(\Gamma^{n-1}_n(E)\ldots))$ for $n > 1$. The set $\Gamma^1_n(E)$ will be called the Γ-projection* of the set E.

By $\mathfrak{A}$ we denote the class formed by those functions $\Phi(z_1, \ldots, z_n, t)$ for which the functions $\Phi(z_1, \ldots, z_n, |z_{n+1}|)$ are plurisubharmonic in C^{n+1}.

We shall also denote, for $\Phi \in \mathfrak{A}$,

$$\Phi^+(z_1, \ldots, z_n, t) = \max\{0, \Phi(z_1, \ldots, z_n, t)\},$$

$$M_\Phi(r_1, \ldots, r_n, t) = \max_{|z_i| \le r_i,\ i=1,\ldots,n} \Phi^+(z_1, \ldots, z_n, t),$$

$$\rho(z_1, \ldots, z_n; \Phi) = \overline{\lim_{t\to\infty}} \frac{\ln \Phi^+(z_1, \ldots, z_n, t)}{\ln t},$$

$$\rho^*(r_1, \ldots, r_n; \Phi) = \overline{\lim_{t\to\infty}} \frac{\ln M_\Phi(r_1, \ldots, r_n, t)}{\ln t},$$

$$\hat\rho = \hat\rho(\Phi) = \sup_{0 \le r_i < \infty,\ i=1,\ldots,n} \rho^*(r_1, \ldots, r_n; \Phi).$$

* For more details on Γ-projections see ([4]).

$$\sigma(z_1, \ldots, z_n; \Phi) = \overline{\lim_{t\to\infty}} \frac{\Phi^1(z_1, \ldots, z_n, t)}{t^{\hat\rho}},$$

$$\sigma^*(r_1, \ldots, r_n; \Phi) = \overline{\lim_{t\to\infty}} \frac{M_\Phi(r_1, \ldots, r_n, t)}{t^{\hat\rho}}.$$

Theorem 1. Let the function $\Phi(z_1, \ldots, z_n, t) \in \mathfrak{A}$, and let the inequality

$$\rho(z_1, \ldots, z_n; \Phi) < \infty$$

hold on a set E satisfying the condition

$$C(\Gamma^1_n(E)) > 0.$$

Then $\hat\rho(\Phi) < \infty$, and the equality

$$\rho(z_1, \ldots, z_n; \Phi) = \hat\rho(\Phi)$$

holds everywhere in C^n, except possibly for some set N_Φ belonging to $G_{\delta\sigma}$ and satisfying the condition: the intersection of N_Φ with any analytic plane

$$\{z_i = a_{iw} + b_i,\ i = 1, \ldots, n\},$$

not contained entirely in N_Φ, has capacity zero.

Theorem 2. Let the function $\Phi(z_1, \dots, z_n, t) \in \mathfrak{A}$, and let the function $M_\Phi(r_1, \dots, r_n, t)$ have order $\rho < \infty$ with respect to the variables jointly*. Suppose further that on some set E satisfying the condition

$$C(\Gamma_n^1(E)) > 0,$$

for some $\gamma < \infty$ the inequality

$$\sigma(z_1, \dots, z_n; \Phi) \leq \gamma$$

holds. Then for every $R > 0$ the inequality

$$\sigma^*(R, \dots, R; \Phi) \leq \gamma C_\Phi R^{\rho - \hat{\rho}},$$

is valid, where C_Φ is a certain constant.

Theorem 3. Let the function $\Phi(z_1, \dots, z_n, t) \in \mathfrak{A}$, let the function $M_\Phi(r_1, \dots, r_n, t)$ have finite order with respect to the variables jointly, and let a set $K \subset C^n$ satisfy the condition

$$C(\Gamma_n^1(K)) > 0,$$

$$\Phi(K, t) \overset{\text{def}}{=} \sup_K \Phi(z_1, \dots, z_n, t).$$

Then, if

$$\int^\infty \frac{\Phi(K, t)}{t^{\hat{\rho}+1}}\, dt < \infty,$$

then

$$\int^\infty \frac{M_\Phi(r_1, \dots, r_n, t)}{t^{\hat{\rho}+1}}\, dt < \infty$$

for all $r_1 \geq 0, \dots, r_n \geq 0$.

Remark 1. As is not hard to see, the class $\mathfrak{A}$ contains the function $\ln M_f(z_1, \dots, z_n, t)$, where the function $M_f(z_1, \dots, z_n, t)$ is defined for an entire function $f(z_1, \dots, z_n, w)$ by the equality

$$M_f(z_1, \dots, z_n, t) = \max_{|w|=t} |f(z_1, \dots, z_n, w)|.$$

Therefore Theorems 1, 2, 3 are valid, with the obvious changes in formulation, also for entire functions.

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* The order ρ of a positive function $\Phi(t_1, \dots, t_m)$ with respect to the variables jointly is defined by the equality

$$\rho = \overline{\lim_{R \to \infty}} \frac{\ln \Phi(R, \dots, R)}{\ln R}.$$

Remark 2. It is not hard to see that the class $\mathfrak{A}$ contains the function

$$\mathcal{L}_f(z_1, \dots, z_n, t) = \int_0^{2\pi} \ln \left| f(z_1, \dots, z_n, e^{i\varphi} t) \right| d\varphi.$$

The functions $\mathcal{L}_f(z_1, \dots, z_n, t)$ and $n_f(t; z_1, \dots, z_n)$ have, with respect to the variable t, one and the same order, type (minimal, normal, or maximal), and belong to the same convergence class. Therefore Theorems 1, 2, 3 remain valid when the function $\Phi \in \mathfrak{A}$ is replaced by the function $n_f(t; z_1, \dots, z_n)$.

Without stopping here to prove the theorems stated above, we shall only point out that we have essentially used the following assertion, which, in our opinion, has independent significance.

Theorem 4. Let E be a closed plane set lying in the disk $|z_1| < R$ and having positive capacity; let E^* be a connected component of the complement of E, containing the point at infinity, and let ∂E^* be the boundary of the domain E^*. Further, let $\omega(z_1)$ be a function harmonic in $E^* \cap (|z_1| < R)$, taking the value 0 everywhere on the circle $|z_1| = R$ and the value 1 at all points $z_1 \in \partial E^*$, except, possibly, for a certain set of capacity zero. Then, if the function $\Phi(z_1, t) \in \mathfrak{A}$* and $\Phi(z_1, T_2) \le A$ for $z_1 \in \partial E^*$, $\Phi(z, T_1) \le B$ for $|z_1| = R$, then for every $z_1 \in E^* \cap (|z_1| \le R)$ the inequality

$$\Phi\left(z_1, T_2^{\omega(z_1)} T_1^{1-\omega(z_1)}\right) \le A\omega(z_1) + B(1 - \omega(z_1)).$$

holds.

Kharkov Aviation
Institute

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CITED LITERATURE

[1] M. Sire, Rend. Circ. Mat. Palermo, **31**, 1 (1911). [2] P. Lelong, Ann. Ec. Norm. Sup., **58**, 83 (1941). [3] L. I. Ronkin, Mat. sborn., **39**, No. 2, 253 (1956); Collection: Investigations on Contemporary Problems of the Theory of Functions of a Complex Variable, 1961, pp. 269–276; DAN, **119**, No. 2, 211 (1958); DAN, **169**, No. 4, 767 (1966). [4] L. I. Ronkin, Mat. sborn., **71**, No. 3 (1966).

* Instead of requiring that the function $\Phi(z_1, t)$ belong to the class $\mathfrak{A}$, it is enough to require that the function $\Phi(z_1, |z_2|)$ be plurisubharmonic in the domain $\{(z_1, z_2);\ z_1 \in E^* \cap (|z_1| < R),\ T_1 < |z_2| < T_2\}$.

Note: Figure translations are in progress. See original paper for figures.