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Abstract

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ON THE THEORY OF THE RADIATION Q-FACTOR OF A DIELECTRIC RESONATOR*

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The effect of total internal reflection leads to the existence of "trapped" types of oscillations in a dielectric resonator even in the absence of conducting walls ([1]). In the approximation of geometrical optics for a transparent dielectric, the attenuation of these oscillations is equal to zero; therefore their Q-factor is determined by radiation from the edges of the resonator. In the case $l \gg \lambda$ (l is the linear dimension of the resonator), the loss powers can be found by knowing the radiation intensity of a plane wave incident on the edge of a rectangular dielectric wedge under conditions of total internal reflection. Below we shall consider this problem for plane geometry in the limiting case of a large refractive index $n^2 \equiv \varepsilon\mu \gg 1$ (ε and μ are the dielectric and magnetic permeabilities of the dielectric; here we have introduced $\mu \neq 1$ only to facilitate the transition from one polarization to the other).

We begin with an H-polarized wave (the magnetic field of the wave is parallel to the edge of the wedge). In the geometrical-optics approximation the fields of this wave inside the wedge ($x < 0, z < 0, -\infty < y < +\infty$) have the form

$$H^{(1)} = \exp(ik_{xx} + ik_{zz}) + R_2 \exp(-ik_{xx} + ik_{zz}) + R_3 \exp(ik_x - ik_{zz}) + R_2 R_3 \exp(-ik_{xx} - ik_{zz});$$
$$k_x \equiv kn\cos\theta; \qquad k_z \equiv kn\sin\theta; \qquad n^2 \equiv \varepsilon\mu; \tag{1}$$
$$R_s \equiv (1-\delta_s)/1+\delta_s; \qquad \delta_2(\theta) \equiv ik_{2\varepsilon}/k; \qquad \delta_3(\theta) \equiv \delta_2(\pi/2-\theta);$$
$$k_2(\theta) \equiv (k_z^2 - k^2)^{1/2}.$$

As is evident from (1), for $n^2 \gg 1$, $\theta \neq 0,\ \pi/2$, to accuracy up to quantities of order n^{-2}, the faces of the wedge may be regarded as perfectly reflecting. Since for a perfectly reflecting wedge the solution (1) is exact, the amplitude of the field scattered into the wedge, arising as a result of the violation of the conditions of applicability of geometrical optics near the edge, is proportional to n^{-2} (the error with which the faces of the wedge may be regarded as perfectly reflecting), and the resulting loss power is proportional to n^{-6}.**

Outside the wedge, the fields corresponding to (1) decrease exponentially in the directions normal to the faces of the wedge and, near the edge, undergo discontinuities in the directions tangent to these faces.

We shall seek the scattered field outside the wedge in the form of the sum of two terms, each of which compensates the discontinuities of the fields (1) penetrating into the vacuum through one of the faces of the wedge. Thus, the field excited by the face

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** The loss power into a wedge with opening angle $\gamma = \pi/m$ ($m = 3, 4, ...$) is of the same order (2). For $\gamma \neq \pi/m$, this power is proportional to n^{-2}.

$x = 0,\ z < 0$, we shall seek in the form

$$\begin{gathered} H^{(2)} = T_2 \exp(ik_z z - k_2 x) + \widetilde{H}^{(2)}, \quad x > 0,\ -\infty < z < +\infty; \\ H^{(3)} = \widetilde{H}^{(3)}, \\ \widetilde{H}^{(s)} \equiv \int_{-\infty}^{+\infty} dt\, h_s(t) \exp[itz - (-1)^s v_s(t)], \quad s = 2, 3,\ \operatorname{Re} v_s(t) > 0; \\ v_s(t) \equiv (t^2 - k^2 n_s^2)^{1/2}; \quad n_s^2 \equiv \varepsilon_s \mu_s; \quad T_2 \equiv 2/(1 + \delta_2). \end{gathered} \tag{2}$$

The unknown Fourier amplitudes $h_s(t)$ are determined from the boundary conditions on the half-planes $z < 0,\ x = 0;\ z > 0,\ x = 0$ and $x < 0,\ z = 0$. From the first two conditions, with the aid of the Wiener-Hopf lemma (3), one can express $h_s(t)$ in terms of the boundary values of functions analytic in the upper (+) and lower (−) half-planes of the complex variable t:

$$\begin{aligned} h_3(t) &= \frac{1}{\Delta_3}\left\{v^+ + Zu^+ - \frac{T_2 k_z}{\pi i}\frac{z_0 - z}{t^2 - k_z^2}\right\}; \\ h_2(t) &= \frac{1}{\Delta_3}\left\{v^+ - Zu^+ - \frac{T_2 k_z}{\pi i}\frac{z_0 + z}{t^2 - k_z^2}\right\} = \\ &= \frac{1}{\Delta_1}\left\{\psi^- - Z_1\varphi^- + \frac{T_2 R_3}{\pi i} k_z \frac{z_0 + Z_1}{t^2 - k_z^2}\right\}; \end{aligned} \tag{3a}$$

$$\begin{gathered} \Delta_s(t) \equiv Z_s(t) + Z(t); \quad Z_s(t) \equiv i v_s(t)/k\varepsilon_s; \quad Z_3 = Z_2 = Z; \\ Z_0 \equiv Z(k)_z. \end{gathered} \tag{3b}$$

The boundary condition on the half-plane $x < 0,\ z = 0$ in the zeroth approximation in n^{-1} has the form $E_x^{(3)}(x < 0,\ z = 0) = 0$ and leads to the relation (4)

$$h_3(t) - h_3(-t) = 0. \tag{4}$$

Adding and subtracting (3b) with its mirror reflection at the point $t = 0$ and substituting (3a) into (4), with the aid of the Sokhotskii-Plemelj formulas [5,6], in the same approximation ($h \gg 1$) we obtain an equation determining the unknown function $v(t) \equiv v^+(t) - v^+(-t)$:

$$\hat{I}v' + \frac{Z}{3}\,\hat{I}\,\frac{v'}{Z'} = F_0 \equiv \frac{2T_2}{3\pi k}k_z^2\frac{1+2R_3}{t^2-k_z^2}, \tag{5}$$

where primes denote functions depending on the integration variable in the operator

$$\hat{I}f' \equiv \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{dt'}{t'-t}\,f(t'),$$

while unprimed functions depend on t.

The index of this equation is zero; therefore its solution, satisfying the condition $v(\infty) = 0$, which ensures finiteness of the field energy near the wedge edge (the Meixner condition), exists and is uniquely determined by the right-hand side. To determine the field in the far zone, according to (2), it is sufficient to know the solution of (5) only in the region $t \le k \le kn$. Setting $v = v_0 + v_1$, where $\hat{I}v_0' \equiv F_0$, we obtain for v_1 equation (5) with right-hand side $F_1 \ll F_0$. From the last inequality it follows that for $t \le kn$ the contribution of v_1 to v may be neglected. Thus, for $t \le kn$ we shall have:

$$v_0^+ = -\frac{T_2 z}{3\pi k};\quad u_0^+ = 0;\quad h_2^{(0)} = h_3^{(0)} = \frac{2T_2}{3\pi k}\frac{1}{Z(t)}. \tag{6}$$

Hence, with the aid of (2), we obtain an expression for the total radiation field in the far zone

$$\widetilde{H}^{(2)} = \widetilde{H}^{(3)} = \frac{8}{3\varepsilon}\frac{1}{\cos 2\psi}\sqrt{\frac{2}{\pi kr}}\exp\left(ikr + \frac{\pi i}{4}\right);\qquad \psi = \pi/4 - \theta;\quad r^2 \equiv x^2 + z^2;\quad kr \gg 1;\quad \varepsilon\cos^2 2\psi \gg 1. \tag{7}$$

The total energy flux of this field S and the resonator quality factor Q determined by it are equal to

$$S = \frac{16}{3\pi}\frac{c\lambda}{\varepsilon^2\cos^2 2\psi};\qquad Q = \frac{3}{128}\frac{\sigma}{\lambda^2}\varepsilon^2\cos^2 2\psi, \tag{8}$$

where σ is the area of the face parallel to the wave vector of the oscillation.

The equations for the scattered fields of an E-polarized wave (with nonzero components E_y, H_z, and H_x) can be obtained from (1)–(3) by making the substitutions $\varepsilon \to \mu$; $\mu \to \varepsilon$; $H \to E$; $E_i \to -H_z$; $E \to -H_x$. The condition analogous to (4) in this case has the form $e_3(t) + e_3(-t) = 0$. The final equation for v, analogous to (5), is

$$\hat{I}v' + \frac{Z}{3}\hat{I}\frac{v'}{Z'} = \frac{2T_2}{\pi k}\frac{k_z^2}{t^2 - k_z^2}, \tag{9}$$

and from its solution in the region $t \ll kn$, $v_0^+ = -T_2/\pi k$, it follows that, in the first approximation in n^{-1}, the scattered field vanishes,

$$\widetilde{E}^{(2)} = \widetilde{E}^{(3)} = 0, \qquad kr \gg 1, \tag{10}$$

so that radiation appears only in the next approximation in this parameter.

The dependence of this field (7) on the refractive index of the wedge n can be explained as follows. Near the wedge in vacuum, the largest amplitude, of order n^{-1}, is possessed by the component of the electric field normal to the surface of the face, corresponding to (1). Therefore the polarization currents compensating the discontinuity of this field component are equivalent to a surface magnetic current, parallel to the edge of the wedge, with density of the order of the discontinuity, i.e., n^{-1}. Since this current flows over a distance of the order of the wavelength in the medium, $\lambda_d \equiv \lambda/n$, the total equivalent magnetic current and its field (7) are proportional to n^{-2}. In the case of an E-polarized wave, the magnetic field undergoes a discontinuity of order unity; therefore the total equivalent electric current is proportional to n^{-1}. Since this current flows on the surface of a dielectric with a large refractive index, which screens this current (the screening coefficient in an unbounded dielectric is equal to n^{-2}, and for a dielectric half-space to $2n^{-1}$), the radiation field, according to (10), is absent in the first approximation in n^{-1} and differs from zero only in the next approximation in n^{-1}.

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Note: Figure translations are in progress. See original paper for figures.