

# ON GLOBAL SOLVABILITY OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ON GLOBAL SOLVABILITY OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

*(Presented by Academician I. G. Petrovskii on 18 XII 1967)*

It is known that, in the case of an arbitrary function  $f(t, u)$ , the Cauchy problem for equation (1) is not globally solvable for every right-hand side  $h(t)$ , i.e., for every time  $t \geq 0$ . The question of how large the set of functions  $h(t)$  is for which global solvability holds has, roughly speaking, the following answer: the codimension of this set in the space  $\mathcal{L}_2(0, \infty)$  is equal to one. This fact is proved in § 1. In § 2 an analogous result is established for an equation of order  $n$ .

§ 1. **Equation of the first order.** Consider on the half-axis the Cauchy problem for the equation

$$u' + f(t, u) = h(t), \quad u(0) = 0. \quad (1)$$

Notation:

$$\overset{0}{H}^{(n)}(0, T) = \left\{ u(t) \mid \int_0^T |u^{(n)}(t)|^2 dt < \infty, u(0) = u(T) = 0, \dots, u^{(n-1)}(0) = u^{(n-1)}(T) = 0 \right\};$$

$$\overset{0}{H}^{(n)}(0, \infty) = \left\{ u(t) \mid \int_0^\infty |u^{(n)}(t)|^2 dt < \infty, u(0) = 0, \dots, u^{(n-1)}(0) = 0 \right\}.$$

**Theorem 1.** Let  $f(t, \xi)$  be differentiable with respect to  $t$  and locally summable with respect to  $\xi$ , and suppose that for any finite function  $u(t)$  on the half-axis

$$\int_0^u \dot{f}(t, u) u' dt \geq -K,$$

where  $K \geq 0$  is a constant.

Then for any function  $h(t) \in \mathcal{L}_2(0, \infty)$  there exist a constant  $C$  and a function  $u(t) \in \overset{0}{H}^{(1)}(0, \infty)$  such that

$$u' + f(t, u) = h(t) + C. \quad (2)$$

In other words, equation (1) is globally solvable up to an additive constant.

**Proof.** Let  $v_1(t), v_2(t), \dots$  be an orthonormal in  $\mathcal{L}_2(0, \infty)$  sequence of smooth functions finite on the half-axis, complete in  $\overset{0}{H}^{(1)}(0, T)$  for every finite  $T$ . (As such a sequence one may take, for example, the union of sequences complete in  $\overset{0}{H}^{(1)}(0, T)$ , where  $T$  is a natural number, and normalize it in  $\mathcal{L}_2(0, \infty)$ .)

We seek an approximate solution  $u_n(t)$  in the form

$$u_n(t) = \sum_{k=1}^n c_{kn} v_k(t),$$

where the constants  $c_{kn}$  are determined from the system of equations

$$\int_0^\infty u_n' v_k' dt + \int_0^\infty f(t, u_n) v_k' dt = \int_0^\infty h v_k' dt, \quad k = 1, \dots, n. \quad (3)$$

The solvability of system (3) follows from a lemma of M. I. Vishik <sup>(1)</sup>; moreover, for the approximate solutions the a priori estimate holds

$$\int_0^\infty |u_n'|^2 dt \leq K \left( \int_0^\infty h^2 dt + 1 \right).$$

This means that there exists a function  $u(t) \in \overset{0}{H}^{(1)}(0, \infty)$  such that some subsequence  $u_s(t) \rightarrow u(t)$  weakly in  $\overset{0}{H}^{(1)}(0, \infty)$ . Since, at the same time, the functions  $u_s(t)$  themselves converge to  $u(t)$  uniformly on any finite interval  $[0, T]$ , it follows from (3) that for any finite function  $v(t)$

$$\int_0^\infty (u' + f(t, u) - h) v' dt = 0,$$

which is equivalent to (2). The theorem is proved.

**Remark 1.** It is clear that the proof may be regarded as an actual process of finding an almost-solution  $u(t)$ .

**Remark 2.** It is also clear that if on some interval  $(a, b)$   $h(t) \in H^{(s)}(a, b)$  and  $f(t, \xi)$  is sufficiently smooth, then  $u(t) \in H^{(s+1)}(a, b)$ . In particular, if  $h(t) \in C^\infty(a, b)$ , then  $u(t) \in C^\infty(a, b)$ .

**Remark 3.** Theorem 1 is valid for any initial condition  $u(0) = u_0$ . For the proof it suffices to put  $w(t) = u(t) - u_0$  and to apply the theorem just proved to the equation  $w' + f(t, w - u_0) = h$ .

**Corollary 1.** Under the conditions of Theorem 1, for any  $h(t) \in \mathcal{L}_2(0, T)$  the Cauchy problem (1) is almost solvable, i.e., there exist  $C$  and  $u(t)$

$$\left( u(0) = 0, \int_0^T |u'|^2 dt < \infty \right)$$

such that (2) holds.

Indeed, let  $h^*(t)$  be the extension by zero of the function  $h(t)$  to the whole half-axis  $[0, \infty)$ . By Theorem 1 there exists an almost-solution  $u^*(t) \in H^{(1)}(0, \infty)$ . Then the restriction of  $u^*(t)$  to the interval  $[0, T]$  gives the answer.

The following theorem on the solvability of the boundary-value problem is also valid.

**Theorem 2.** Under the conditions of Theorem 1, for any  $h(t) \in \mathcal{L}_2(0, T)$  there exist a constant  $C$  and a function  $u(t) \in H^{(1)}(0, T)$  such that

$$u' + f(t, u) = h(t) + C.$$

§ 2. **The general case.** Consider on the half-axis  $[0, \infty)$  the Cauchy problem

$$u^{(n)} + f(t, u, \dots, u^{(n-1)}) = h(t), \quad (4)$$

$$u(0) = 0, \dots, u^{(n-1)}(0) = 0. \quad (5)$$

**Theorem 3.** Let  $f(t, \xi_0, \dots, \xi_{n-1})$  be locally summable in each variable, and for any finite function  $u(t)$  on the half-axis

$$\int_0^\infty f(t, u, \dots, u^{(n-1)}) u^{(n)} dt \geq -K, \quad (6)$$

where  $K \geq 0$  is a constant.

Then for any  $h(t) \in \mathcal{L}_2(0, \infty)$  there exists a polynomial

$$P_{n-1}(t) = C_0 + C_1 t + \dots + C_{n-1} t^{n-1}$$

and a function  $u(t) \in \overset{0}{H}^{(n)}(0, \infty)$  such that

$$u^{(n)} + f(t, u, \dots, u^{(n-1)}) = h(t) + P_{n-1}(t). \quad (7)$$

The proof of this theorem repeats the proof of Theorem 1.

**Remark 4.** Simple examples show that, if condition (6) is violated, Theorem 3, generally speaking, does not hold.

**Corollary 2.** Under the conditions of Theorem 3, for any  $h(t) \in \mathcal{L}_2(0, T)$  there exist a polynomial  $P_{n-1}(t)$  and a function  $u(t)$  ( $u(0) = 0, \dots, u^{(n-1)}(0) = 0$ ,

$$\int_0^T |u^{(n)}|^2 dt < \infty)$$

such that (7) holds.

**Corollary 3.** If on some interval  $(a, b)$ ,  $h(t) \in H^{(s)}(a, b)$  and  $f(t, \xi_0, \dots, \xi_{n-1})$  is sufficiently smooth, then  $u(t) \in H^{(s+n)}(a, b)$ . In particular, for  $h(t) \in C^\infty(a, b)$  the almost-solution  $u(t) \in C^\infty(a, b)$ .

**Theorem 4 (boundary-value problem).** Under the conditions of Theorem 3, for any  $h(t) \in \mathcal{L}_2(0, T)$  there exists a polynomial  $P_{n-1}(t)$  such that equation (7) is solvable in  $\overset{0}{H}^{(n)}(0, T)$ .

We give the simplest examples of equations for which the conditions of the preceding theorems are fulfilled.

1.

$$u' + a_0(u) = h(t).$$

2.

$$u'' + a_0(u') + a_1(u) = h(t).$$

3.

$$u^{(n)} + a_0(u^{(n-1)}) + a_1(u^{(n-2)}) + a_2(u^{(n-3)}) + \dots + a_n u = h(t).$$

Here the function  $a_0(\xi)$  is arbitrary,  $a_1(\xi)$  is a nonincreasing function,  $a_k$  ( $k \geq 2$ ) are numbers, with  $(-1)^m a_{2m} \geq 0$ , while  $a_{2m+1}$  are arbitrary.

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## REFERENCES

1. M. I. Vishik, DAN, **137**, No. 3, 502 (1961).

*Note: Figure translations are in progress. See original paper for figures.*

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