ON THE PROBLEM WITH AN INITIAL JUMP FOR NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS CONTAINING A SMALL PARAMETER

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Abstract

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MATHEMATICS

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ON THE PROBLEM WITH AN INITIAL JUMP FOR NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS CONTAINING A SMALL PARAMETER

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§ 1. Suppose we have the Cauchy problem for a system of ordinary differential equations with a small parameter multiplying the highest derivative

$$\varepsilon dz/dt = F(z,y), \qquad dy/dt = G(z,y), \tag{1}$$

$$y(t)\big|_{t=0} = y_0^0, \qquad z(t)\big|_{t=0} = z_0^0/\varepsilon^{1/(m+1-n)} \quad (z_0^0 > 0), \tag{2}$$

where $\varepsilon > 0$ is a small parameter; $n-1 < m < n$; $n \geqslant 1$, and $F(z,y)$ and $G(z,y)$, as $|z| \to \infty$, grow respectively like $|z|^n$ and $|z|^m$; for simplicity of exposition we assume that the functions $F(z,y)$ and $G(z,y)$ are represented in the form

$$F(z,y) = z^n\left[f_0(y) + \sum_{i>0} f_i(y)z^{-i}\right], \qquad G(z,y) = z^m\left[g_0(y) + \sum_{i>0} g_i(y)z^{-i}\right], \tag{3}$$

$$f_0(y) < 0, \qquad g_0(y) > 0. \tag{4}$$

The solution of the nondegenerate system of equations (1), as $\varepsilon \to 0$, will tend to the solution of the degenerate system corresponding to (1):

$$0 = F(z,y), \qquad dy/dt = G(z,y), \qquad y(t)\big|_{t=0} = y_0^0 + \Delta y_0^0, \tag{5}$$

where the quantity Δy_0^0 will be called the **initial jump** of the function $y(t)$. Let us now find the length of the initial jump Δy_0^0. Since in the jump zone the independent variable t changes little, while y changes by a finite amount close to Δy_0^0, system (1), in view of (3), may be replaced by the system

$$\varepsilon dz/dt = f_0(y)z^n, \qquad dy/dt = g_0(y)z^m, \tag{6}$$

whence

$$\varepsilon \frac{dz}{dy} = \frac{f_0(y)}{g_0(y)} z^{n-m}. \tag{7}$$

Solving equation (7), we have:

$$z^{m+1-n} = z^{m+1-n} - (m+1-n) \int_{y_0^0}^{y_0^0+\Delta y_0^0} \frac{f_0(y)}{g_0(y)}\, dy \,/\, \varepsilon, \tag{8}$$

where z and z denote, respectively, the initial and final values of the function $z(t)$. If we assume that $z = O(1)$, then from (8) we obtain

$$z = \left[-(m+1-n) \int_{y_0^0}^{y_0^0+\Delta y_0^0} \frac{f_0(y)}{g_0(y)}\, dy \,/\, \varepsilon \right]^{1/(m+1-n)}. \tag{9}$$

Comparing now formula (9) with the initial condition (2), we find an expression for computing the length of the initial jump Δy_0^0

$$(z_0^0)^{m+1-n} = -(m+1-n) \int_{y_0^0}^{y_0^0+\Delta y_0^0} \frac{f_0(y)}{g_0(y)}\, dy. \tag{10}$$

To construct an asymptotic expansion of the solution of problem (1), (2), the region of the boundary layer, where the solution of the nondegenerate problem differs substantially from the solution of the degenerate problem, is divided into two zones (see (1, 2)), and a third zone—the zone of limited variation of the solution, where the solution of the nondegenerate problem already falls into a small neighborhood of the solution of the degenerate problem. In the present paper the asymptotics is constructed only in the first two zones.

§ 2. In the first zone we take the variable y as the independent variable and, instead of the system of equations (1), we shall consider the equation

$$\varepsilon\, dz/dy = F(z, y)/G(z, y), \tag{11}$$

and instead of (2) the initial condition:

$$z(y)\big|_{y=y_0^0} = z_0^0/\varepsilon^{1/(m+1-n)}. \tag{12}$$

The substitution

$$u(y) = \varepsilon^s z(y), \qquad s = 1/(m+1-n) \tag{13}$$

reduces problem (11), (12) to the problem

$$\frac{du}{dy} = \frac{u^n\left[f_0(y) + \varepsilon^s f_1(y)u^{-1} + \varepsilon^{2s} f_2(y)u^{-2} + \cdots\right]}{u^m\left[g_0(y) + \varepsilon^s g_1(y)u^{-1} + \varepsilon^{2s} g_2(y)u^{-2} + \cdots\right]}, \tag{14}$$

$$u(y)\big|_{y=y_0^0} = z_0^0. \tag{15}$$

We seek the solution of problem (14), (15) in the form of a formal series in ε^s

$$u(y) = u_0(y) + \varepsilon^s u_1(y) + \varepsilon^{2s} u_2(y) + \cdots. \tag{16}$$

Substituting expansion (16) into (14), (15) and equating the coefficients of equal powers of ε^s, we obtain a sequence of differential equations for determining the coefficients $u_i(y)$ $(i \geq 0)$ in (16):

$$\frac{du_0}{dy} = \frac{f_0(y)}{g_0(y)} u_0^{n-m}, \qquad u_0(y_0^0) = z_0^0, \tag{17}$$

$$\frac{du_i}{dy} - (n-m)\frac{f_0(y)}{g_0(y)} u_0^{n-m-1} u_i = \Phi_i(y, u_0, \dots, u_{i-1}), \qquad u_i(y_0^0) = 0. \tag{18}$$

The following estimates hold near $\bar{y}$ $(y < \bar{y} = y_0^0 + \Delta y_0^0)$:

$$u_i(y) = O\left[1/(\bar{y} - y)^{(i-1)/(m+1-n)}\right] \qquad (i \geq 0). \tag{19}$$

Assume that the first zone ends at $y = y_1^0$, where $y_1^0 < \bar{y}$:

$$y - y_1^0 = O(\varepsilon^\sigma), \qquad 0 < \sigma < 1. \tag{20}$$

In this zone we shall prove the following proposition, which shows to what degree the computed functions $u_i(y)$ represent the exact solution of problem (14), (15).

Theorem 1. Every solution $u(y, \varepsilon)$ of differential equation (14), satisfying the initial condition (15), in the first zone $y_0^0 \leq y \leq y_1^0 < \bar{y}$ has the asymptotic representation

$$u(y,\varepsilon) = \bar{u}_N(y,\varepsilon) + R_N(y,\varepsilon), \qquad \bar{u}_N(y,\varepsilon) = \sum_{i=0}^{N} \varepsilon^{is} u_i(y), \tag{21}$$

$$|R_N(y,\varepsilon)| < C_N \varepsilon^{s(N+1)}/(\bar{y}-y)^{sN}, \tag{22}$$

where C_N is a certain positive constant independent of ε.

For the proof, take two curves:

$$u(y,\varepsilon) = \bar{u}_N(y,\varepsilon) + \varepsilon^{s(N+1)}\omega(y), \tag{23}$$

$$u(y,\varepsilon) = \bar{u}_N(y,\varepsilon) - \varepsilon^{s(N+1)}\omega(y). \tag{24}$$

Here by $\omega(y)$ we denote the function

$$\omega(y) = M_N/(\bar{y}-y)^{sN}, \tag{25}$$

where M_N is a certain sufficiently large constant independent of ε. We compute the derivative by virtue of equations (14) on the curve (23). We have

$$\begin{aligned} T_{(23)}(y) = {} & [\bar{u}_N(y,\varepsilon) + \varepsilon^{s(N+1)}\omega(y)]^m \left[g_0(y) + \sum_{i>0} \varepsilon^{is} g_i(y)(\bar{u}_N + \varepsilon^{s(N+1)}\omega(y))^{-i} \right] \times \\ & \times (d\bar{u}_N/dy + \varepsilon^{s(N+1)} d\omega/dy) - [\bar{u}_N(y,\varepsilon) + \varepsilon^{s(N+1)}\omega(y)]^n \times \\ & \times \left[f_0(y) + \sum_{i>0} \varepsilon^{is} g_i(y)(\bar{u}_N + \varepsilon^{s(N+1)}\omega(y))^{-i} \right]. \end{aligned} \tag{26}$$

If the right-hand side of equality (26) is expanded in powers of ε^s, then all terms of order ε^{si} $(0 \le i \le N)$ vanish and, consequently, expression (26) is equal to zero up to a quantity of order ε^{sN}. Therefore, for sufficiently large M_N, the sign of the function $T_{(23)}(y)$ is determined by the term that is positive, and consequently $T_{(23)}(y) > 0$. In an analogous manner one may verify that the derivative by virtue of equation (14) on the curve (24) is negative, i.e. $T_{24}(y) < 0$. Thus every solution of equation (14) under conditions (15) passes between the curves (23) and (24). Hence estimate (22) follows. In particular, at the end of the zone $y = y_1^0$ estimate (22) takes the form $R_N = O(\varepsilon^{s(N+1-N\sigma)})$. The theorem is proved.

It is straightforward to verify that the value $y = y_1^0$ corresponds to

Fig. 1

Figure 1: Fig. 1

$$t = t_1^0 = \begin{cases} O(\varepsilon|\ln\varepsilon^\sigma|), & n = 1, \\ O(\varepsilon^{\sigma_1}), & n > 1, \end{cases} \qquad z = z_1^0 = O\left(\frac{1}{\varepsilon^{s(\sigma-1)}}\right), \qquad \sigma_1 = \frac{m - \sigma(n-1)}{m+1-n} > 1.$$

§ 3. In the second zone the Cauchy problem for the system of differential equations (1) is solved under the initial conditions

$$y(t)\big|_{t=t_1^0} = y_1^0, \qquad z(t)\big|_{t=t_1^0} = z_1^0. \tag{27}$$

By the substitution $\tau = (t - t_1^0)/\varepsilon$, problem (1), (27) reduces to the problem

$$dz/d\tau = F(z,y), \qquad z(\tau)\big|_{\tau=0} = z_1^0; \qquad dy/d\tau = \varepsilon G(z,y), \qquad y(\tau)\big|_{\tau=0} = y_1^0. \tag{28}$$

Solving problem (28) in the first approximation, taking (4) into account, we obtain

$$z(t) = \begin{cases} O(z_1^0)\exp(-(t-t_1^0)/\varepsilon), & \text{for } n = 1, \\ O\left[\dfrac{1}{\left((t-t_1^0)/\varepsilon + 1/(z_1^0)^{n-1}\right)^{1/(n-1)}}\right], & \text{for } n > 1. \end{cases} \tag{29}$$

Taking z in this zone as the independent variable, from (28) we obtain

$$dy/dz = \varepsilon G(z,y), F(z,y), \qquad y(z)\big|_{z=z_1^0} = y_1^0. \tag{30}$$

We shall seek the solution $y(z,\varepsilon)$ of problem (30) in the form of an expansion in powers of the small parameter ε:

$$y(z,\varepsilon) = y_0(z) + \varepsilon y_1(z) + \varepsilon^2 y_2(z) + \dots, \qquad y_0(z) \equiv y_1^0. \tag{31}$$

Substituting now expansion (31) into (30) and equating coefficients of like powers of ε, we obtain the following sequence of equations for $y_i(z)$, $i > 0$:

$$dy_i/dz = H_i(z, y_0, y_1, \dots, y_{i-1}), \qquad y_i(z_1^0) = 0. \tag{32}$$

Fig. 1

The solution of equation (32) has an estimate of the form:

$$y_i(z) = O\left[(z_1^0)^{i(m+1-n)}\right], \qquad i > 0. \tag{33}$$

Let the second zone now end at $t = t_2^0$, where

$$t_2^0 = \begin{cases} O(\varepsilon|\ln\varepsilon|), & \text{if } n = 1, \\ O(\varepsilon), & \text{if } n > 1. \end{cases} \tag{34}$$

Then it follows from formula (29) that $z(t)$ at $t = t_2^0$ becomes a finite quantity. Thus, in the second zone the variable z changes from a quantity of order $1/\varepsilon^{s(1-\sigma)}$ to some finite quantity z_2^0. Analogously to the first zone, one proves:

Theorem 2. Every solution $y(z,\varepsilon)$ of the differential equation (30) in the second zone $z_1^0 \le z \le z_2^0$ admits the asymptotic representation

$$y(z,\varepsilon) = y_0(z) + \varepsilon y_1(z) + \ldots + \varepsilon^N y_N(z) + S_N(z,\varepsilon), \tag{35}$$

$$|S_N(z,\varepsilon)| < M_N \varepsilon^{(N+1)\sigma}, \tag{36}$$

where M_N is a positive constant independent of ε.

Since at the end of the second zone the quantity z already becomes finite, the further behavior of the integral curve of problem (1), (2) is studied by the method of ([3]).

Remark. The investigations presented above carry over directly to the case when the variable quantity y is vector-valued.

Appendix. As an application, consider the boundary-value problem

$$\varepsilon y'' = \varphi(t, y, y'), \qquad y(0) = 0, \qquad y(1) = 0. \tag{37}$$

For $\varepsilon = 0$ we obtain the degenerate problem

$$\varphi(t, y, y') = 0, \qquad y(1) = 0 \tag{38}$$

(the degenerate equation can also be solved at the point $t = 0$). Denote by $y(t,\varepsilon)$ the solution of the nondegenerate problem (37), and by $\overline{y}(t)$ the solution of the degenerate problem (38). Let $y(t,\varepsilon) \to \overline{y}(t)$ as $\varepsilon \to 0$. The resulting geometric picture is presented in Fig. 1 ($1-\overline{y}(t)$, $2-y(t,\varepsilon)$). From the drawing it is seen that $y'(0,\varepsilon) = g(\varepsilon) \to \infty$ as $\varepsilon \to 0$ and $y(t,\varepsilon) \to \overline{y}(t)$, $\varepsilon \to 0$, for all $0 < t_2^0(\varepsilon) \le t \le 1$, $t_2^0(\varepsilon) \to 0$, $\varepsilon \to 0$; the quantity $\overline{y}(0) - y(0,\varepsilon) = \Delta y_0^0$ is called the initial jump of the function $y(t)$. Obviously, the order of growth of the function $g(\varepsilon)$ as $\varepsilon \to 0$ depends on the order of growth of the function $\varphi(t, y, y')$ with respect to y'.

Thus, in order to solve the boundary-value problem (37), one must first solve the Cauchy problem with an initial jump for (37), i.e.

$$\varepsilon y'' = \varphi(t, y, y'), \qquad y(0) = 0, \qquad y'(0) = g(\varepsilon) \to \infty,\ \varepsilon \to 0. \tag{39}$$

In the proposed article precisely the problem of the form (39), or the problem of the form (39) for an equation of higher order, is considered.

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Note: Figure translations are in progress. See original paper for figures.