ON THE FLOW PAST BODIES OF REVOLUTION BY A SONIC STREAM OF A VISCOUS HEAT-CONDUCTING GAS

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Abstract

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ON THE FLOW PAST BODIES OF REVOLUTION BY A SONIC STREAM OF A VISCOUS HEAT-CONDUCTING GAS

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An exact solution of the approximate equations for near-sonic flow of a viscous heat-conducting gas (^{1}) has been found in the form of a functional series in inverse powers of the distance from the axis of the flow. It is represented in quadratures and contains a countable set of arbitrary constants. For the problem of flow past a finite body of revolution, the solution obtained makes it possible to establish the principal qualitative feature of the solution of the Navier–Stokes equations—the formation behind the body in the flow of a vortex wake. Within the framework of the indicated approximate equations (^{1}), the main part of the correction to the asymptotic solution (^{2}), valid at very large distances from the body, is computed. Analogous problems for the Kármán equations in the theory of near-sonic flow of an ideal gas were considered in (^{3,4}). A theory of the second approximation for the full Navier–Stokes equations is the subject of (^{5}).

1. The approximate equations of steady near-sonic flow of a viscous heat-conducting gas in the axisymmetric case have the form (^{1})

$$\partial^2 v_x/\partial x^2 + \partial v_r/\partial r + v_r/r = v_x \partial v_x/\partial x, \qquad \partial v_r/\partial x = \partial v_x/\partial r. \tag{1,1}$$

Here x and r denote dimensionless cylindrical coordinates, and v_x and v_r are the corresponding dimensionless components of the perturbation-velocity vector.

In what follows we shall assume that, for the problem of near-sonic flow past a finite body of revolution, equations (1,1) model the Navier–Stokes equations, and on this basis, by studying the solutions of these simpler equations, we shall attempt to establish the principal qualitative features of the solutions of the complete equations.

Proceeding from the results of (^{2}), we shall seek the solution of the nonlinear system of differential equations (1,1), describing the perturbations introduced by a finite body of revolution into an oncoming sonic gas stream at infinity, in the form

$$v_x = \sum_{N=1}^{\infty} v_{xN} = r^{-4/3} f_1(\xi) + r^{-2} f_2(\xi) + \dots + r^{-\frac{2}{3}(1+N)} f_N(\xi) + \dots ,$$

$$(\xi = x/r^{2/3}) \qquad (1,2)$$

$$v_r = \sum_{N=1}^{\infty} v_{rN} = r^{-5/3} g_1(\xi) + r^{-7/3} g_2(\xi) + \dots + r^{-\frac{1}{3}(3+2N)} g_N(\xi) + \dots$$

Substituting the expansions (1,2) into equations (1,1) and equating terms with identical powers of r, and eliminating from the equations thus obtained the functions $g_N(\xi)$, we arrive at ordinary differential equations of the third order for the functions $f_N(\xi)$

$$\frac{d^3 f_N}{d\xi^3} + \frac{4}{9}\xi^2 \frac{d^2 f_N}{d\xi^2} + \frac{4}{9}(3+2N)\xi \frac{d f_N}{d\xi} + \frac{4}{9}(1+N)^2 f_N = \frac{d q_N}{d\xi},$$

$$q_N = \frac{1}{2}\frac{d}{d\xi}\sum_{n=1}^{N-1} f_n f_{N-n}, \qquad N = 1, 2, 3, \dots \qquad (1,3)$$

Having solved equation (1.3), the function $g_N(\xi)$ can then be found from the relation

$$g_N = \frac{3}{2N}\left[\frac{d^2 f_N}{d\xi^2} + \frac{4}{9}\xi^2 \frac{d f_N}{d\xi} + \frac{4}{9}(1+N)\xi f_N - q^N\right]. \qquad (1.4)$$

The nonlinear term appearing on the right-hand side of equation (1.3) does not depend on the function $f_N(\xi)$ itself, but is expressed in terms of the preceding functions $f_1(\xi)$, $f_2(\xi), \dots, f_{N-1}(\xi)$. Thus, equation (1.3) turns out to be a linear nonhomogeneous equation with an already known right-hand side, while the corresponding homogeneous equation coincides with the equation obtained for $f_N(\xi)$ upon substituting the expansions (1.2) into the system of equations (1.1) without the nonlinear term on the right-hand side of the first of them and eliminating the function $g_N(\xi)$. This considerably simplifies the problem and, in particular, makes it possible to use the results of [2].

For the function $f_1(\xi)$, (1.3) yields the homogeneous equation

$$\frac{d^3 f_1}{d\xi^3} + \frac{4}{9}\xi^2 \frac{d^2 f_1}{d\xi^2} + \frac{20}{9}\xi \frac{d f_1}{d\xi} + \frac{16}{9} f_1 = 0, \qquad (1.5)$$

two linearly independent particular solutions of which, according to [2], have the form

$$f_{11} = \Psi({}^{2}\!/_{3},\ {}^{1}\!/_{3};\ -{}^{4}\!/_{27}\xi^3), \qquad f_{12} = e^{-{}^{4}\!/_{27}\xi^3}\Psi(-{}^{1}\!/_{3},\ {}^{1}\!/_{3};\ {}^{4}\!/_{27}\xi^3). \tag{1.6}$$

Here the Ψ-functions are solutions of the degenerate hypergeometric equation [6].

Let us now find the general solution of equation (1.5). As was shown in [2], two particular solutions of the corresponding homogeneous equation for any number N can be obtained by differentiating N times with respect to ξ the two particular solutions (1.6) of equation (1.5):

$$f_{N1} = \frac{d^N}{d\xi^N}\left[\Psi({}^{2}\!/_{3},\ {}^{1}\!/_{3};\ -{}^{4}\!/_{27}\xi^3)\right],$$

$$f_{N2} = \frac{d^N}{d\xi^N}\left[\exp(-{}^{4}\!/_{27}\xi^3)\Psi(-{}^{1}\!/_{3},\ {}^{1}\!/_{3};\ {}^{4}\!/_{27}\xi^3)\right]. \tag{1.7}$$

Knowing these two particular solutions of the homogeneous equation, the general solution of equation (1.3) can be represented in the form [7]

$$f_N = c_{N1}f_{N1} + c_{N2}f_{N2} + c_{N3}f_{N3} + f_{N*},$$

$$f_{N3} = f_{N2}\int f_{N1}W_N^{-2}\exp({}^{4}\!/_{27}\xi^3)\,d\xi - f_{N1}\int f_{N2}W_N^{-2}\exp({}^{4}\!/_{27}\xi^3)\,d\xi,$$

$$f_{N*} = f_{N2}\int f_{N1}W_N^{-2}\exp({}^{4}\!/_{27}\xi^3)\left(\int W_N\frac{dq_N}{d\xi}\,d\xi\right)d\xi- \tag{1.8}$$

$$- f_{N1}\int f_{N2}W_N^{-2}\exp({}^{4}\!/_{27}\xi^3)\left(\int W_N\frac{dq_N}{d\xi}\,d\xi\right)d\xi,$$

$$W_N = \left(f_{N1}\frac{df_{N2}}{d\xi} - f_{N2}\frac{df_{N1}}{d\xi}\right)\exp({}^{4}\!/_{27}\xi^3).$$

Here f_{N3} is the third linearly independent solution of the homogeneous equation; f_{N*} is a particular solution of the nonhomogeneous equation; c_{N1}, c_{N2}, c_{N3} are arbitrary constants.

2. To solve the problem of the flow past a finite body of revolution by a sonic gas flow as $x \to -\infty$, we must satisfy the homogeneous boundary conditions

$$
\begin{aligned}
&v_{xN} \to 0 \quad \text{as } r \to 0, \quad x \to -\infty, \\
&v_{rN} = 0 \quad \text{as } r = 0, \quad x < 0.
\end{aligned}
\tag{2.1}
$$

The condition that the components of the disturbed velocity decay for large values of the radius r is satisfied automatically by the choice of the solution in the form (1.2) and by the requirement that the functions $f_N(\xi)$, $g_N(\xi)$ be bounded.

To satisfy the boundary conditions (2.1), let us write out the asymptotic laws of behavior of the functions $f_{N1}(\xi)$, $f_{N2}(\xi)$, $f_{N3}(\xi)$ for large negative values of ξ [6]

$$
\begin{aligned}
f_{N1} &= \xi^{-(N+1)} \left[(-1)^{N+1} \frac{9}{2\sqrt[3]{2}} N! + (-1)^{N+1} \frac{9}{8\sqrt[3]{2}} (N+3)! \xi^{-3} + O(\xi^{-6}) \right], \\
f_{N2} &= \xi^{2N-1} \exp\left(-\frac{4}{27} \xi^3 \right) \left[-\left(2\sqrt[3]{4}/3 \right) \left(-\frac{4}{9} \right)^{N-1} + O(\xi^{-3}) \right], \\
f_{N3} &= \xi^{-(N+1)} \ln|\xi| \left[(3/4\sqrt[3]{2})(9/4)^N \cdot 1/N! + O(\xi^{-3}) \right].
\end{aligned}
\tag{2.2}
$$

For $N = 1$ equation (1.3) is homogeneous and the function $f_{1*}(\xi)$ is identically equal to zero. As is seen from (2.2), the function $f_{12}(\xi)$ far from the body is proportional to $\exp(-\frac{4}{27}\xi^3)$. Therefore, by virtue of the first condition (2.1), we must take $c_{12} = 0$. From the second condition (2.1), taking account of relations (1.2), (1.4), and (2.2), it follows that also $c_{13} = 0$.

Analogous reasoning can be carried out for any number N. First let us establish only the asymptotic law of variation of the function $f_{N*}(\xi)$ as $\xi \to -\infty$. Substituting the dependences (2.2) into the expression for $f_{N*}(\xi)$ from (1.8) and using the method of complete mathematical induction, for $\xi \to -\infty$ we have

$$
f_{N*} = \xi^{-(N+4)} \left\{ \frac{1}{4}(N+2)(N+3)A_N + O(\xi^{-3}) \right\} \left(A_N = \frac{1}{2} \sum_{n=1}^{N-1} c_{n1} c_{N-n,1} \right). \tag{2.3}
$$

At the same time, from the boundary conditions (2.1) it follows that

$$
c_{N2} = 0, \qquad c_{N3} = 0, \qquad N = 1, 2, 3, \dots \tag{2.4}
$$

3. Let us investigate the behavior of the flow on its axis of symmetry $r = 0$. Upstream the variable x is negative and, correspondingly, $\xi \to -\infty$.

Using the representations (2.2), (2.3) together with equalities (2.4), we arrive at the asymptotic laws of behavior of the components of the perturbation-velocity vector as $r \to 0$, $x < 0$

Fig. 1

Figure 1: Fig. 1

$$
\begin{gathered}
v_x = (9/2\sqrt[3]{2})\left[c_{11}x^{-2} - 2c_{21}x^{-3} + \cdots + (-1)^{N+1}N!c_{N1}x^{-(N+1)} + \cdots\right] + O(r^2/x^5),\\
v_r = -\Big\{(27/2\sqrt[3]{2})c_{11}x^{-4} + \left[-(54/\sqrt[3]{2})c_{21} + c_{11}^2\right]x^{-5} + \cdots\\
\cdots + [(N+2)/2]\left[(-1)^{N+1}(9/2\sqrt[3]{2})(N+1)!c_{N1} + A_N\right]x^{-(N+3)} + \cdots\Big\} + O(r^3/x^7). \qquad (3.1)
\end{gathered}
$$

On the left part of the axis of symmetry the transverse component of the velocity vector v_r vanishes, which corresponds to the second condition of (2.1).

The right part of the flow axis ($r = 0,\ x > 0$) corresponds to infinitely large positive values of the variable ξ.

Finding asymptotic representations for the functions $f_{N1}(\xi),\ f_{N2}(\xi)$ [6], and then also for $f_{N*}(\xi)$ as $\xi \to +\infty$, and substituting the dependences obtained into formulas (1.8), (1.2), we write the leading terms of the asymptotic expansions for the components v_x and v_r near the flow axis behind the body being streamlined ($r \to 0,\ x > 0$)

$$
\begin{gathered}
v_x = -\frac{3}{2}\left[W_2^0x^{-3} + W_3^0x^{-4} + \cdots + W_N^0x^{-(N+1)} + \cdots\right]\ln r + O(1/x^2),\\
v_r = \frac{3}{4}\left[W_2^0x^{-2} + W_3^0x^{-3} + \cdots + W_N^0x^{-N} + \cdots\right]r^{-1} + O(r/x^4), \qquad (3.2)\\
W_N^0 = (3\sqrt[3]{2}N!)^{-1}(9/4)^N\int_{-\infty}^{\infty} W_N\,(dq_N/d\xi)\,d\xi.
\end{gathered}
$$

Knowing the explicit form of the expressions $W_N(\xi)$ and $q_N(\xi)$ from (1.3), (1.8) and the asymptotic representations of the functions $f_N(\xi)$, one can show that the improper integral in the last expression converges and the quantity W_N^0 is finite.

As follows from formulas (3.2), on the axis of the flow behind the body being streamlined, singularities arise in the solution (1.8) ($v_x \sim \ln r,\ v_r \sim 1/r$). Therefore, in some neighborhood of the axis the solution obtained is unsuitable. This is explained by the formation, behind a body streamlined by a viscous heat-conducting gas, of a vortex wake, where equations (1.1) cannot be used. This qualitative result is connected only with the nonlinearity of the original equations (1.1), which is also confirmed in the theory of the second approximation for the complete Navier–Stokes equations (^{5}).

Fig. 1

Fig. 2

Figure 2: Fig. 2

The solution found for the problem of flow around a body makes it possible to determine the boundary of the vortex wake. Let us replace the vortex wake by a semi-infinite body of revolution with generatrix $R = R(x)$. The boundary condition on this body is represented in the form

$$\lim_{r \to 0} r v_r = R(dR/dx). \tag{3.3}$$

Substituting the quantity v_r from (3.2) into (3.3) and integrating the resulting equation, we find the law of variation of the cross-sectional area of the vortex wake σ along the x-axis

$$\sigma = \pi R^2 = \sigma_0 - (3\pi/2)\{W_2^0 x^{-1} +$$

$$+ (W_3^0/2)x^{-2} + \cdots$$

$$\cdots + [W_N^0/(N-1)]x^{-(N-1)} + \cdots\}. \tag{3.4}$$

Here σ_0 is an integration constant, having the meaning of the cross-sectional area of a semi-infinite cylinder, the flow around which, if the nonlinear term in the first equation (1.1) is not taken into account, is equivalent to the flow around a source of finite intensity placed in a sound flow of a viscous heat-conducting gas (^{2}).

Fig. 2

4. The results obtained make it possible, in particular, to calculate the next term of the asymptotic expansion of the solution of equations (1.1). To this end we take $c_{11} = -3/2^{3/2}$, $c_{N1} = 0$ ($N = 2, 3, ...$). Under these assumptions calculations were carried out and the functions $f_{2*}(\xi)$ and, correspondingly, $g_{2*}(\xi)$, representing the principal part of the correction to the asymptotic solution, were found. The results of the calculations are presented in Figs. 1 and 2, where, for comparison, the graphs of the functions $f_1(\xi)$ and $g_1(\xi)$ are also given. The correction to the asymptotic solution for the complete Navier–Stokes equations was obtained in (^{5}).

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