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Abstract

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ON AN ABSTRACT TRIANGULAR REPRESENTATION OF LINEAR BOUNDED OPERATORS AND THE MULTIPLICATIVE DECOMPOSITION OF THE CORRESPONDING CHARACTERISTIC FUNCTIONS

(Presented by Academician S. L. Sobolev on 25 X 1967)

Let $\mathfrak{H}$ be a separable Hilbert space and $\mathfrak{R} = \mathfrak{R}_{\mathfrak{H}}$ the ring of linear bounded operators acting in $\mathfrak{H}$. A completely continuous operator $B \in \mathfrak{R}$ will be assigned to the class $\mathfrak{S}_\omega$ ([1,2]) if the eigenvalues $s_j(B)$ of the nonnegative operator $(B^*B)^{1/2}$, numbered in decreasing order with multiplicities counted, satisfy the condition

$$B_{\mathfrak{S}_\omega} = \sum_j \frac{s_j(B)}{2j-1} < \infty.$$

The introduction of the norm $\|\cdot\|_{\mathfrak{S}_\omega}$ turns the class $\mathfrak{S}_\omega$ into a symmetrically normed ideal* of the ring $\mathfrak{R}$. In the present article, operators $A \in \mathfrak{R}$, the imaginary components of which belong to $\mathfrak{S}_\omega$, are reduced to an abstract triangular form, and the corresponding characteristic functions—to a multiplicative one. This generalizes the results of one of the authors ([4,5]), pertaining to the case when A has a purely real spectrum. Another approach to the problem of triangular representation of operators of the class considered by us is contained in the works of L. de Branges ([6,7]).

1. Let us specify a closed chain of orthoprojectors $\mathfrak{P} = \{P\}$ in $\mathfrak{H}$ and functions $F_1(P)$, $F_2(P)$, $G(P)$ ($P \in \mathfrak{P}$) with values in $\mathfrak{R}$. We define the integral

   $$\int_{\mathfrak{P}} F_2(P)\, dG(P)\, F_1(P)$$

   as the limit in Shatunovskii's sense (in the sense of one topology or another) of integral sums of the form

   $$\sum_{j=1}^{m} F_2(Q_j)(G(P_j) - G(P_{j-1}))F_1(Q_j) \tag{1}$$

$$(\min \mathfrak{P} = P_0 < P_1 < \cdots < P_m = \max \mathfrak{P},\ P_{j-1} \leq Q_j \leq P_j;\ P_j, Q_j \in \mathfrak{P}).$$

With the additional restriction $P_{j-1} < Q_j$ or $P_{j-1} = Q_j$, we shall write instead of $\int_{\mathfrak{P}}$ respectively $\int_{(\mathfrak{P}]}$ or $\int_{[\mathfrak{P}}$.

If the chain $\mathfrak{P}$ is maximal and is an increasing sequence $\{P_j\}_0^\infty$, then, obviously: a) $P_0 = 0$, $P_\infty = I$; b) $\dim\{P_j \mathfrak{H}\} = j$, c) $P_j \to I$. Such a chain will be called *discrete*.

Let $\mathfrak{P} = \{P_j\}_0^\infty$ be a maximal discrete chain and let $\varphi(P)$ $(P \in \mathfrak{P})$ be a bounded scalar function. It is easy to verify that then

$$\int_{(\mathfrak{P}]} \varphi(P)\, dP = \sum_{j=1}^{\infty} \varphi(P_j) \Delta P_j \quad (\Delta P_j = P_j - P_{j-1}). \tag{2}$$

The integral and the series in (2) converge strongly.

* The authors adhere to the terminology adopted in the monographs of I. Ts. Gokhberg and M. G. Krein ([2,3]).

Lemma 1. *If the operator $A \in \mathfrak{R}$ has a maximal discrete chain $\mathfrak{P} = \{P_j\}_0^\infty$, then*

$$A = \int_{[\mathfrak{P}]} \alpha(P)\, dP + 2i \int_{[\mathfrak{P}]} P A_I\, dP \left(A_I = \frac{A - A^*}{2i} \right), \tag{3}$$

where the values $\alpha(P_j)$ of the scalar function $\alpha(P)$ are determined from the relations $\alpha(P_j)\Delta P_j = \Delta P_j A \Delta P_j$. The integrals in the equality (3) converge strongly. In the case when $A_I \in \mathfrak{S}_\omega$, the integral

$$\int_{[\mathfrak{P}]} P A_I\, dP$$

converges uniformly.

Proof. For given $h \in \mathfrak{H}$ and $\varepsilon > 0$ choose a natural number $N = N(h, \varepsilon)$ so that the inequality $\|(I - P_N)h\| < \varepsilon / 2\|A\|$ is satisfied. Any continuation of the partition $P_0 < P_1 < \cdots < P_N < I$ has the form $Q_0 < Q_1 < \cdots < Q_n = I$, where $Q_j = P_j$ $(j = 1, 2, \ldots, N)$. Since

$$A = \sum_{j=1}^{n} \Delta Q_j A \Delta Q_j + 2i \sum_{j=1}^{n} Q_{j-1} A_I \Delta Q_j \quad (\Delta Q_j = Q_j - Q_{j-1})$$

and, consequently,

$$\left\|(A-\hat{A})h-2i\sum_{j=1}^{n}Q_{j-1}A_I\Delta Q_j h\right\|^2=\left\|\sum_{j=N+1}^{n}\Delta Q_j A\Delta Q_j h-\sum_{j=N+1}^{n}\Delta P_j A\Delta P_j h\right\|^2$$
$$\le(2\|A\|)^2\|(I-P_N)h\|^2<\varepsilon^2$$

$$\left(\hat{A}=\sum_{j=1}^{\infty}\alpha(P_j)\Delta P_j=\sum_{j=1}^{\infty}\Delta P_j A\Delta P_j=\int_{[\mathfrak{P}]}\alpha(P)\,dP\right),$$

the integral

$$\int_{[\mathfrak{P}]}PA_I\,dP$$

exists in the sense of strong convergence. At the same time relation (3) has been proved.

If $A_I\in\mathfrak{S}_\omega$, then $A_I-\hat{A}_I\in\mathfrak{S}_\omega$, where

$$\hat{A}_I=\sum_{j=1}^{\infty}\Delta P_j A_I\Delta P_j$$

(see (2), p. 74). Moreover, evidently,

$$\Delta P_j(A_I-\hat{A}_I)\Delta P_j=0\quad(j=1,2,\ldots).$$

It follows from this (see (3), p. 132) that, in the sense of uniform convergence, the integral

$$\int_{\mathfrak{P}}P(A_I-\hat{A}_I)\,dP$$

exists. In particular, for a given $\varepsilon>0$ there is a partition of the chain $\mathfrak{P}$ such that any continuation of it $0=P_0'<P_1'<\cdots<P_k'=I$ will satisfy the inequality

$$\left\|\int_{\mathfrak{P}}P(A_I-\hat{A}_I)\,dP-\sum_{j=1}^{k}P_{j-1}'(A_I-\hat{A}_I)\Delta P_j'\right\|<\varepsilon.$$

Since

$$P_{j-1}'\hat{A}_I\Delta P_j'=0\quad(j=1,2,\ldots,k),$$

the integral

$$\int_{[\mathfrak{P}]}PA_I\,dP$$

converges uniformly.

The lemma is proved. Consider an operator $A\in\mathfrak{R}$ with an imaginary component from $\mathfrak{S}_\omega$. Denote by $\mathfrak{H}_0$ the closure of the linear span of its root subspaces corresponding to nonreal points of the spectrum, and introduce the orthoprojectors $P^{(0)}$ and $P^{(1)}$ respectively onto $\mathfrak{H}_0$ and $\mathfrak{H} \ominus \mathfrak{H}_0$. Let A_0 be the operator induced in $\mathfrak{H}_0$, and A_1 the operator in $\mathfrak{H}_1$ defined by the formula

$$A_1 h = P^{(1)} A h \quad (h \in \mathfrak{H}_1).$$

The operator A_0 has a maximal discrete chain $\mathfrak{P}^{(0)} = \{Q_j\}_0^\infty$*, and the operator A_1** has such a maximal chain–

* We consider, for definiteness, only the case when $\dim \mathfrak{H}_0 = \infty$.
** The whole spectrum of the operator A_1 lies on the real axis ([8]).

for which $\mathfrak{P}^{(1)} = \{Q\}$, such that for every real t there is an orthoprojector $Q_t \in \mathfrak{P}^{(1)}$ cutting its spectrum at t ([4]). Define the functions $\alpha_0(Q)$ $(Q \in \mathfrak{P}^{(0)})$ and $\alpha_1(Q)$ $(Q \in \mathfrak{P}^{(1)})$ by setting

$$\alpha_0(Q_j)\Delta Q_j = \Delta Q_{jA} \Delta Q_j, \qquad \alpha_1(Q) = \max \sigma[A_1/Q],$$

where $\sigma[A_1/Q]$ is the spectrum of the restriction of the operator QA_1Q to the subspace $Q\mathfrak{H}_1$.

Theorem 1. *If $A \in \mathfrak{R}$ and $A_I \in \mathfrak{S}_\omega$, then*

$$A = \int_{\mathfrak{P}} \alpha(P)\, dP + 2i \int_{[\mathfrak{P}]} P A_{IdP}, \tag{4}$$

where $\mathfrak{P}$ is a maximal chain in $\mathfrak{H}$, which is the union of the chains $\mathfrak{P}_0 = \{QP^{(0)}\}$ $(Q \in \mathfrak{P}^{(0)})$ and $\mathfrak{P}_1 = \{P^{(0)} + QP^{(1)}\}$ $(Q \in \mathfrak{P}^{(1)})$,

$$\alpha(P) = \begin{cases} \alpha_0(Q), & P = QP^{(0)} \qquad (Q \in \mathfrak{P}^{(0)}), \\ \alpha_1(Q), & P = P^{(0)} + QP^{(1)} \qquad (Q \in \mathfrak{P}^{(1)}). \end{cases}$$

The first integral in (4) converges strongly, and the second uniformly.

Proof. By Lemma 1 and the main theorem of ([4]),

$$P^{(0)} A P^{(0)} = A_0 P^{(0)} = \int_{\mathfrak{P}_0} \alpha(P)\, dP + 2i \int_{[\mathfrak{P}_0]} P A_{IdP}, \tag{5}$$

$$P^{(1)} A P^{(1)} = A_1 P^{(1)} = \int_{\mathfrak{P}_1} \alpha(P)\, dP + 2i \int_{\mathfrak{P}_1} P^{(1)} P A_{IdP}. \tag{6}$$

The integrals in (6) converge uniformly. From the relations $P^{(0)} A P^{(1)} = 2iP^{(0)} A_{IP}^{(1)}$, $(P^{(0)} A P^{(1)})^2 = 0$ it follows that the operator $P^{(0)} A P^{(1)}$ is Volterra. Since it possesses the chain $\mathfrak{P}$, in the sense of uniform convergence ([3],[9]),

$$P^{(0)}AP^{(1)} = \int_{\mathfrak{P}} PP^{(0)}AP^{(1)}\,dP = 2i\int_{\mathfrak{P}} PP^{(0)}P^{(1)}\,dP = 2i\int_{\mathfrak{P}_1} P^{(0)}PA_{IdP}. \quad (7)$$

The assertion of the theorem follows from (5), (6), (7) and the equality

$$A = P^{(0)}AP^{(1)} + P^{(1)}AP^{(1)} + P^{(0)}AP^{(1)}.$$

Formula (4) shows that an operator $A \in \mathfrak{R}$ with imaginary component from $\mathfrak{S}_\omega$ can be represented as the sum of a normal operator and a Volterra operator possessing one and the same maximal chain.

2. Let us specify an operator node $\theta = \begin{pmatrix} A & K & J \\ \mathfrak{H} & \mathfrak{G} & \end{pmatrix}$ and consider its characteristic operator-function ([10], [11])

$$W_\theta(\lambda) = I - 2iK^*(A - \lambda E)^{-1}KJ.$$

Lemma 2. Let $\mathfrak{S}$ be an arbitrary symmetrically normed ideal of the ring $\mathfrak{R}_{\mathfrak{G}}$. If $K^*K \in \mathfrak{S}$ and $T \in \mathfrak{R}_{\mathfrak{H}}$, then $K^*TK \in \mathfrak{S}$ and

$$\|K^*TK\|_{\mathfrak{S}} \leq 2\|T\|\,\|K^*K\|_{\mathfrak{S}}.$$

Lemma 3. If A possesses a discrete maximal chain $\mathfrak{P} = \{P_j\}_0^\infty$ and the operator K^*K belongs simultaneously to the symmetrically normed ideals $\mathfrak{S}_\omega$ and $\mathfrak{S}$, then

$$W_\theta(\lambda) = \prod_{j=1}^{\infty}\left(I + 2i\,\frac{K^*\Delta P_{jKJ}}{\lambda - \lambda_j}\right), \qquad (\Delta P_{jA}\Delta P_j = \lambda_j \Delta P_j), \qquad (8)$$

where the infinite product converges in the norm of the ideal $\mathfrak{S}$.

Let an operator $J \in \mathfrak{R}_{\mathfrak{G}}$ satisfy the conditions $J = J^*$, $J^2 = I$. An operator-function of the complex variable $W(\lambda)$, whose values belong to $\mathfrak{R}_{\mathfrak{G}}$, will be assigned to the class Ω_J if: 1) $W(\lambda)$ is holomorphic in the domain G_W obtained by deleting from the extended complex plane some bounded set having no non-real limit points; 2) $W(\infty) = I$; 3) $W^*(\lambda)JW(\lambda) - J \geq 0$ ($\operatorname{Im}\lambda > 0,\ \lambda \in G_W$); 4) $W^*(\lambda)JW(\lambda) - J = 0$ ($\operatorname{Im}\lambda = 0,\ \lambda \in \overline{G_W}$);

5) all operators $W(\lambda) - I$ ($\lambda \in G_W$) are completely continuous. From 3) and 4) it follows that in a neighborhood of the infinitely distant point the function $W(\lambda)$ expands into a series of the form

$$W(\lambda) = I + \frac{2i}{\lambda} H_{WJ} + \ldots \qquad (H_W \geqslant 0).$$

Theorem 2*. Let $W(\lambda) \in \Omega_J$. If the operator H_W belongs simultaneously to the symmetrically normed ideals $\mathfrak{S}_\omega$ and $\mathfrak{S}$, then

$$W(\lambda) = \prod_{j=1}^{n} \left(I + \frac{2i}{\lambda - \lambda_j} F_{jJ} \right) \int_0^1 \left(I + \frac{2i}{\lambda - \alpha(x)} dF(x) J \right) \qquad (n \leqslant \infty), \quad (9)$$

where λ_j are nonreal numbers; $\alpha(x)$ is a scalar function, continuous from the left and nondecreasing; F_j are one-dimensional positive operators satisfying the condition $F_{jJF}j = \operatorname{Im} \lambda_{jF} j$; $F(x)$ $(0 \leqslant x \leqslant 1)$ is a positive, strictly increasing, absolutely continuous operator-function whose values belong to the ideal $\mathfrak{S}$. The integral products

$$\prod_{k=1}^{m} \left(I + \frac{2i}{\lambda - \alpha(\xi_k)} \Delta F_{kJ} \right)$$

$$(0 = x_0 < x_1 < \ldots < x_m = 1, \ x_{k-1} < \xi_k \leqslant x_k, \quad \Delta F_k = F(x_k) - F(x_{k-1}))$$

converge, in the sense of Shatunovskii, in the norm of the ideal $\mathfrak{S}$. In the sense of the same norm there converges the product of the factors $I + \frac{2i}{\lambda - \lambda_j} F_{jJ}$ when $n = \infty$.

Proof. The function $W(\lambda)$ is the characteristic function for some simple node

$$\theta = \begin{pmatrix} AKJ \\ \mathfrak{H} \ \mathfrak{G} \end{pmatrix}$$

with completely continuous operator K [5, 10]. Moreover, $W_\theta(\lambda) = W_0(\lambda) W_1(\lambda)$, where $W_0(\lambda)$ and $W_1(\lambda)$ are the characteristic operator-functions of the projections of the node θ onto the subspaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$, introduced in the first section of the article. Applying to the functions $W_0(\lambda)$ and $W_1(\lambda)$, respectively, Lemma 3 of the present article and Theorem 1 of article [5], we obtain formula (9). The convergence of the multiplicative integral in the sense of the norm of the ideal $\mathfrak{S}$ follows from Lemma 2.

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* Multiplicative representations of functions of the class Ω_J ($H_W \in \mathfrak{S}_1$), where $\mathfrak{S}_1$ is the ideal of all nuclear operators, were studied in detail by Yu. P. Ginzburg by purely analytic methods. In particular, for this case he established Theorem 2 as well ([12,13]).

Note: Figure translations are in progress. See original paper for figures.