# ON SPACES OF CLOSED SUBSETS

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## Abstract

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**MATHEMATICS**

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# ON SPACES OF CLOSED SUBSETS

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Let $X$ be a topological space. Denote by $\exp X$ the new topological space whose points are the nonempty closed subsets $F$ of the space $X$, and whose open base consists of the sets

$$\langle U_1, \dots, U_n \rangle = \mathcal{E}\left( F \in \exp X : \ F \subseteq \bigcup_{k=1}^{n} U_k, \ F \cap U_k \neq \emptyset, \ k = 1, 2, \dots, n \right),$$

where $U_1, \dots, U_n$ is an arbitrary finite collection of nonempty open sets of the space $X$. Numerous properties of the space $\exp X$ are described in detail in ($^{2-5}$). It should be noted that the space $\exp X$, endowed with the indicated topology, is always a $T_0$-space. If, however, $X$ is normal, then $\exp X$ is completely regular ($^2$). Moreover, the conditions that $X$ is (bi)compact and that $\exp X$ is (bi)compact are equivalent ($^3$).

In this note the connection between the weight characteristics of the spaces $X$ and $\exp X$ is investigated, a topological description of $\exp X$ is given for certain spaces $X$, and, finally, the connection between the dimensions of $X$ and $\exp X$ is studied.

**Lemma 1.** *The system*

$$\mathfrak{B} = \{\langle U_{\alpha_1}, \dots, U_{\alpha_n} \rangle\},$$

*whose elements are all possible nonempty finite collections of sets of the open base*

$$\mathfrak{b} = \{U_\alpha\}$$

*of the bicompactum $X$, forms an open base of the bicompactum* $\exp X$*.*

**Proof.** Let $F \in \langle G_1, \dots, G_n \rangle \subseteq \exp X$. Since $G_1, \dots, G_n$ are open in $X$, and $\mathfrak{b}$ is a base in $X$, we have

$$G_k = \bigcup_{\xi} U^{\xi}_{\alpha(k)}, \quad k = 1, 2, \dots, n.$$

From the definition of the topology in $\exp X$ it follows that in $X$

$$F \subseteq \bigcup_{k=1}^{n} \bigcup_{\xi} U_{\alpha(k)}^{\xi}; \tag{1}$$

$$F \cap G_k = F \cap \left( \bigcup_{\xi} U_{\alpha(k)}^{\xi} \right) = \bigcup_{\xi} \left( F \cap U_{\alpha(k)}^{\xi} \right) \neq \emptyset, \quad k = 1, 2, \dots, n. \tag{2}$$

For each $k = 1, 2, \dots, n$ choose from the covering $\{U_{\alpha(k)}^{\xi}\}$ of the bicompactum $F \subseteq X$ those $U_{\alpha(k)}^{\xi}$ for which $F \cap U_{\alpha(k)}^{\xi} \neq \emptyset$. As (2) shows, such $U_{\alpha(k)}^{\xi}$ exist for every $k = 1, 2, \dots, n$ and together form an open covering of $F$. Choose from this covering a finite one. If, however, there is a $k$, $1 \leq k \leq n$, such that not a single $U_{\alpha(k)}^{\xi} \subseteq G_k$ participates in the obtained covering, then adjoin to the covering an arbitrary element $U_{\alpha(k)}^{\xi}$ for which $F \cap U_{\alpha(k)}^{\xi} \neq \emptyset$. We obtain a covering

$$\{U_{\alpha(1)}^{1}, \dots, U_{\alpha(1)}^{p_1}, \dots, U_{\alpha(n)}^{1}, \dots, U_{\alpha(n)}^{p_n}\}$$

of the bicompactum $F$, where

$$U_{\alpha(k)}^{i} \subseteq G_k, \quad k = 1, 2, \dots, n, \quad i = 1, 2, \dots, p_k. \tag{3}$$

Obviously,

$$F \in \langle U_{\alpha(1)}^{1}, \dots, U_{\alpha(1)}^{p_1}, \dots, U_{\alpha(n)}^{1}, \dots, U_{\alpha(n)}^{p_n} \rangle.$$

By virtue of (3) one obtains the inclusion

$$\langle U_{\alpha(1)}^{1}, \dots, U_{\alpha(1)}^{p_1}, \dots, U_{\alpha(n)}^{1}, \dots, U_{\alpha(n)}^{p_n} \rangle \subseteq \langle G_1, \dots, G_n \rangle.$$

The lemma is proved.

A system $\sigma = \{U_\alpha\}$ of open sets of the space $X$ is called **dense** (Ponomarev) in this space if, for every open $G \subseteq X$, there exists $U_\alpha \in \sigma$ such that $U_\alpha \subseteq G$.

**Lemma 2.** The system $\Sigma = \{\langle U_{\alpha_1}, \dots, U_{\alpha_n} \rangle\}$, whose elements are nonempty finite collections of sets from the system $\sigma = \{U_\alpha\}$, dense in the space $X$, is dense in $\exp X$.

The least cardinal number that is the cardinality of an open base of the space $X$ is called the **weight** of the space $X$ and is denoted by $wX$.

**Theorem 1.** The weight of the space $\exp X$ coincides with the weight of $X$, if $X$ is a bicompactum and $wX \geq \aleph_0$.

The least cardinal number that is the cardinality of a system of open sets dense in $X$ is called the **$\pi$-weight** of the space $X$ and is denoted by $w_\pi X$ (Ponomarev).

**Theorem 2.** For an arbitrary topological space $X$,

$$w_\pi X = w_\pi(\exp X), \quad \text{if } w_\pi X \geq \aleph_0.$$

Let $\mathfrak{B}_x = \{O_\alpha x\}$ be some fundamental system of neighborhoods of a point $x \in X$. The cardinal number $\inf_\alpha(wO_\alpha x)$ is called the **local weight of the point $x$ in the space $X$.**

The space $X$ is called **homogeneous with respect to local weight** if $w(x, X)$ is constant on $X$ (6).

**Theorem 3.** The bicompactum $\exp X$ is homogeneous with respect to local weight and $w(F, \exp X) = \tau$ for every point $F \in \exp X$, if $X$ is homogeneous with respect to local weight and $w(x, X) = wX = \tau \geq \aleph_0$.

**Proof.** A bicompactum $X$ is homogeneous with respect to local weight and $w(x, X) = wX = \tau$ if and only if for every point $x \in X$ and any of its neighborhoods $Ox$ there exists a bicompactum $B_x$ such that $x \in B_x \subseteq Ox$, $wB_x = \tau$.

Take a point $F \in \exp X$ and any of its neighborhoods $\langle U_1, \dots, U_n\rangle$ from the base of $\exp X$. Since $\exp X$ is a bicompactum, there exists a smaller neighborhood $\langle V_1, \dots, V_s\rangle$ of the point $F$, for which

$$F \in \langle V_1, \dots, V_s\rangle \subseteq \langle \overline{V_1}, \dots, \overline{V_s}\rangle = \langle \overline{V}_1, \dots, \overline{V}_s\rangle \subseteq \langle U_1, \dots, U_n\rangle.$$

In each $V_i$, $i = 1, 2, \dots, s$, there is contained a bicompactum $B_i$ of weight $\tau$; hence $w\overline{V}_i = \tau$. To obtain a base $\mathfrak{B}$ of the bicompactum $\langle \overline{V}_1, \dots, \overline{V}_s\rangle$, one must in each $\exp \overline{V}_i$ choose a base $\mathfrak{B}_i$ from sets of the form $\langle U^i_{\alpha_1}, \dots, U^i_{\alpha_n}\rangle$, $i = 1, 2, \dots, s$, where $\mathfrak{b}_i = \{U^i_\alpha\}$ is a base of $\overline{V}_i$, and unite them over $i = 1, 2, \dots, s$ into common brackets:

$$\langle U^1_{\alpha_1}, \dots, U^1_{\alpha_{p(1)}}, \dots, U^s_{\omega_1}, \dots, U^s_{\omega_{p(s)}}\rangle \in \mathfrak{B}.$$

If $|\mathfrak{B}_i| = \tau$ *, $i = 1, 2, \dots, s$, then $|\mathfrak{B}| = \tau$, and therefore

$$w\langle \overline{V}_1, \dots, \overline{V}_s\rangle \leq \tau.$$

Since all $V_i$ are distinct, among them there will be some $V_i$, and in it a bicompactum $B_i$, such that $B_i \cap V_j = \emptyset$ for $i \neq j$, $wB_i = \tau$. The bicompactum $\exp B_i$ is topologically contained in $\langle \overline{V}_1, \dots, \overline{V}_s\rangle$. Since

$$wB_i = w(\exp B_i) = \tau,$$

it follows that

$$w\langle \overline{V}_1, \dots, \overline{V}_s\rangle = \tau.$$

The theorem is proved.

The **neighborhood character** $\chi(F, X)$ of a set $F$ in the space $X$ is the minimum of the cardinalities of fundamental systems of neighborhoods of $F$ in $X$ ($^7$). The space $X$ is called **homogeneous with respect to neighborhood character** if $\chi(x, X)$ is constant on $X$ ($^6$). The **pseudocharacter** $\psi(F, X)$ of a set $F$ in $X$ is the least number $\mathfrak{m}$ such that

$$F = \bigcap_{\alpha \in A} O_\alpha,$$

where all $O_\alpha$ are open and $|A| = \mathfrak{m}$ ($^7$).

It is known that $\psi(F, X) = \chi(F, X)$ if $F$ is closed and $X$ is a bicompactum.

**Theorem 4.** For an arbitrary topological space $X$ and a closed $F \subseteq X$,

$$\chi(F, X) \leq \tau, \quad \text{if } \chi(F, \exp X) \leq \tau.$$

In ($^7$) the following problem is formulated: does there exist a bicompactum with the first axiom of countability, having cardinality $> \mathfrak{c}$, where $\mathfrak{c}$ is the cardinality of the continuum?

**Theorem 5.** Any bicompactum $Y = \exp X$ satisfying the first axiom of countability has cardinality $\leq \mathfrak{c}$.

* The symbol $|A|$ denotes the cardinality of the set $A$.

**Proof.** Assuming the contrary, by means of Theorem 4 we conclude that the bicompactum $X$ is perfectly normal*. In that case $|X| \leq \mathfrak{c}$ (7). Since in $X$ there are as many closed sets as open sets, and every open set, being of type $F_\sigma$, is the sum of no more than a countable number of closed sets, it follows that $|\exp X| \leq \mathfrak{c}$. The contradiction obtained proves the theorem.

A **caliber** of a space $X$ is a cardinal number $\mathfrak{m}$ such that every system of nonempty open sets of the space $X$ of cardinality $\mathfrak{m}$ has an equicardinal subsystem that has a nonempty intersection (Shanin).

It is known that every regular cardinal number $\mathfrak{m} > \aleph_0$ is a caliber of any dyadic bicompactum (6).

**Theorem 6.** *A regular cardinal number* $\mathfrak{m} > \aleph_0$ *is a caliber of the space* $\exp X$ *if and only if* $\mathfrak{m}$ *is a caliber of* $X$.

**Proof.** We shall carry it out for a system of sets of an open base of the space $\exp X$. Let

$$\Sigma = \{\langle U_{\alpha_1}, \dots, U_{\alpha_s} \rangle\}$$

be a system of nonempty open sets in $\exp X$ of regular cardinality $\mathfrak{m} > \aleph_0$. It is assumed that the order of the elements in the brackets is fixed. The **length** of a set

$$\langle U_{\alpha_1}, \dots, U_{\alpha_s} \rangle \in \Sigma$$

is the number $s > 0$. By virtue of the regularity of the number $\mathfrak{m}$, there is a number $n > 0$ such that the system $\Sigma$ contains an equicardinal subsystem $\Sigma_n^{(0)}$, all elements of which have the same length $n$. From each

$$\langle U_{\alpha_1}, \dots, U_{\alpha_n} \rangle \in \Sigma_n^{(0)}$$

the first element $U_{\alpha_1}$ is selected, and the system

$$\sigma_1 = \{U_{\alpha_1}\}$$

of the selected sets is considered in the space $X$. Since $\mathfrak{m}$ is a caliber of $X$, there exists

$$\sigma_1' \subseteq \sigma_1, \qquad |\sigma_1'| = |\sigma_1| = \mathfrak{m},$$

having a nonempty intersection in $X$. In $\exp X$ we consider the system

$$\Sigma_n^{(1)} \subseteq \Sigma_n^{(0)},$$

for whose elements $U_{\alpha_1} \in \sigma_1'$ in $X$. With the system $\Sigma_n^{(1)}$ the same is done as with $\Sigma_n^{(0)}$, but with respect to the second elements $U_{\alpha_2}$. As a result one obtains a system

$$\Sigma_n^{(2)} \subseteq \Sigma_n^{(1)}.$$

Continuing the process described, at the $n$-th step we obtain a system

$$\Sigma_n^{(n)} \subseteq \Sigma_n^{(n-1)},$$

all $n$-th elements of whose sets belong to the system

$$\sigma_n' = \{U_{\alpha_n}\},$$

which has a nonempty intersection in $X$. Thus:

$$\Sigma \supseteq \Sigma_n^{(0)} \supseteq \Sigma_n^{(1)} \supseteq \dots \supseteq \Sigma_n^{(n)}; \tag{4}$$

$$\left|\Sigma_n^{(k)}\right| = |\Sigma| = \mathfrak{m}, \qquad k = 0, 1, \dots, n; \tag{5}$$

$$\Sigma_n^{(n)} = \{\langle U_{\alpha_1}, \dots, U_{\alpha_n} \rangle\}, \qquad U_{\alpha_k} \in \sigma_k', \quad k = 1, 2, \dots, n. \tag{6}$$

In the space $X$, for each $k = 1, 2, \dots, n$, choose a point

$$x_k \in \bigcap_{\sigma_k'} U_{\alpha_k}.$$

Then

$$F = \{x_1, \dots, x_n\} \in \exp X,$$
$$F \in \bigcap_{\Sigma_n^{(n)}} \langle U_{\alpha_1}, \dots, U_{\alpha_n} \rangle.$$

Hence, and from (4), (5), the necessity of the condition follows. For the proof of sufficiency only the formula

$$\bigcap_{\alpha}\langle U_{\alpha}\rangle = \left\langle \bigcap_{\alpha} U_{\alpha} \right\rangle$$

is needed (2).

A topological space $X$ satisfies the **Suslin condition** if every system of disjoint open sets from $X$ is at most countable (see (6)).

**Corollary 1.** *The spaces $X$ and $\exp X$ simultaneously satisfy the Suslin condition if $\aleph_1$ is a caliber of one of them.*

**Corollary 2.** *The space $X$ satisfies the Suslin condition if the space $\exp X$ satisfies it.*

* A space $X$ is called **perfectly normal** if it is normal and each of its closed sets is of type $G_{\delta}$.

A topological space $X$ satisfies the **Knaster condition** if every uncountable system of nonempty open subsets of $X$ contains an uncountable subsystem every pair of whose sets has nonempty intersection (see (9)).

**Theorem 7.** *The topological space $\exp X$ satisfies the Knaster condition if and only if $X$ satisfies this condition.*

The **density** $sX$ of a space $X$ is the least cardinality of everywhere dense subsets of the space $X$.

**Theorem 8.** *The densities of the spaces $X$ and $\exp X$ coincide.*

Denote by $T_{\tau}$ the discrete $T_1$-space of cardinality $\tau \geq \aleph_0$, and by $\mathfrak{b}_0 T_{\tau}$ its one-point compactification with vertex $\mathfrak{b}_0$ (the Alexandrov space) (see (1)).

**Theorem 9.** *The bicompactum $\exp \mathfrak{b}_0 T_{\tau}$ is a compactification of $\mathfrak{b} T_{\tau}$ with remainder $\mathfrak{b} T_{\tau} \setminus T_{\tau}$, homeomorphic to $D^{\tau}$* for every $\tau \geq \aleph_0$.

**Proof.** Denote

$$X = \mathcal{E}(F \in \exp \mathfrak{b}_0 T_{\tau} : \mathfrak{b}_0 \notin F), \quad Y = \mathcal{E}(F \in \exp \mathfrak{b}_0 T_{\tau} : \mathfrak{b}_0 \in F).$$

Obviously, $X \cup Y = \exp \mathfrak{b}_0 T_{\tau}$, $X \cap Y = \emptyset$, $X$ is open, and $Y$ is closed in $\exp \mathfrak{b}_0 T_{\tau}$. It is not hard to see that $X$ is homeomorphic to $T_{\tau}$. Renumber the points $t \in T_{\tau}$:

$$T_{\tau} = \{t_{\alpha}\}_{\alpha \in A} |A| = \tau$$

and represent

$$D = \prod_{\alpha \in A} D_{\alpha}^{(0,1)}.$$

The mapping $f : Y \to D^{\tau}$ consists in assigning to each point $F \in Y$ the point $x \in D^{\tau}$ with coordinates

$$x_{\alpha} = 1, \quad \text{if } t_{\alpha} \in F \subseteq \mathfrak{b}_0 T_{\tau}, \qquad x_{\alpha} = 0, \quad \text{if } t_{\alpha} \notin F.$$

It is easy to see that $f$ is a homeomorphism onto all of $D^\tau$.

Let $\aleph_0 \leq \mathfrak{m} < \tau$. The **$\Sigma\mathfrak{m}$-product** is the set

$$\Sigma D^\tau \subset D^\tau$$

consisting of those points $x = \{x_\alpha\} \in D^\tau$ for which $x_\alpha \neq 0$ for at most a set of indices of cardinality $\leq \mathfrak{m}$ (6). Denote

$$\exp_{\mathfrak{m}} X = \mathcal{E}(F \in \exp X : |F| \leq \mathfrak{m})$$

for any cardinal number $\mathfrak{m} \geq \aleph_0$.

**Corollary.** *In* $\exp \mathfrak{b}_0 T_\tau$,

$$(\exp \mathfrak{b}_0 T_\tau \setminus T_\tau) \cap \exp_{\mathfrak{m}} \mathfrak{b}_0 T_\tau \cong \Sigma_{\mathfrak{m}} D^\tau.$$

The dimension of the space $Y$ sometimes allows one to judge whether $Y$ is representable in the form $\exp X$.

**Theorem 10.** *The compactum* $\exp X$ *is infinite-dimensional if* $\dim X \geq 1$.

**Proof.** The mapping

$$f_n : X^n \to \exp_n X$$

is (4) open-and-closed and $n!$-to-one. Therefore

$$\dim X^n = \dim(\exp_n X).$$

Using results from (8), it is easy to obtain the inequality, for any $n < \aleph_0$:

$$\dim(\exp_n X) = \dim X^n \geq n, \quad \text{if } \dim X \geq 1.$$

Since $\exp_n X$ is closed in $\exp X$ for every $n = 1, 2, \ldots$ and $\dim(\exp_n X) \geq n$, the space $\exp X$ is an infinite-dimensional compactum.

**Corollary.** The space

$$\exp_{\aleph} X = \mathcal{E}(F \in \exp X : |F| < \aleph_0)$$

of all finite subsets of the compactum $X$ is weakly infinite-dimensional.

It remains an open question: will $\exp X$ be a strongly infinite-dimensional compactum if $\dim X \geq 1$?

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$^{*}$ $D^{\tau}$

is the generalized Cantor set, i.e. the Tikhonov product of $\tau$ discrete two-point spaces, $\tau \geq \aleph_0$.

*Note: Figure translations are in progress. See original paper for figures.*

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