ON THE EXISTENCE OF MOMENTS OF THE NUMBER OF LEVEL CROSSINGS BY A GAUSSIAN STATIONARY PROCESS

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Abstract

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ON THE EXISTENCE OF MOMENTS OF THE NUMBER OF LEVEL CROSSINGS BY A GAUSSIAN STATIONARY PROCESS

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It is known ([1,2]) that the m-th factorial moment of the number of crossings from below upward of the level a by a Gaussian process without linear relations ξ_t on the interval Δ is equal to

$$d_m(\Delta, a) = \int_{\{t_i \in \Delta,\ t_i \neq t_j,\ i,j=1,\dots,m\}} M\left\{\prod_{i=1}^{m} \dot{\xi}^{+}_{t_i} \mid \xi_{t_j} = a,\ j = 1, \dots, m\right\} \times P_{t_1 \dots t_m}(a, \dots, a)\, dt_1 \dots dt_m, \tag{1}$$

where $P_{t_1 \dots t_m}(x_1, \dots, x_m)$ is the density of the joint distribution of the quantities $\xi_{t_1}, \dots, \xi_{t_m}$ with covariance matrix $R_{11}(t_1, \dots, t_m)$,

$$\dot{\xi}_{t_i} = d\xi_{t_i}/dt_i, \qquad \dot{\xi}^{+}_{t_i} = \tfrac{1}{2}(\dot{\xi}_{t_i} + |\dot{\xi}_{t_i}|).$$

Below we study the question of conditions for the existence of the factorial moments specified by the integrals (1). Denote

$$I(l) = \int_{\{t_i \in \Delta,\ t_i \neq t_j,\ i,j=1,\dots,m\}} \frac{\mu_{i_1} \dots \mu_{i_l} \sigma_{i_{l+1}} \dots \sigma_{i_m}}{\sqrt{|R_{11}(t_1, \dots, t_m)|}}\, dt_1 \dots dt_m,$$

where

$$\mu_i = M\{\dot{\xi}_{t_i} \mid \xi_{t_j} = a,\ j = 1, \dots, m\}, \qquad \sigma_i^2 = D\{\dot{\xi}_{t_i} \mid \xi_{t_j} = a,\ j = 1, \dots, m\},$$

$(i_1, \dots, i_m)$ is a permutation of the indices $(1, \dots, m)$.

Theorem 1. *For the existence of the m-th moment of the number of crossings of the level a by a Gaussian process it is sufficient that the integrals $I(l)$ ($l = 0, \dots, m$) exist, and necessary that the integral $I(0)$ exist.*

Sufficiency was proved by Yu. K. Belyaev ([1]), Lemma 4. Necessity follows from two lemmas.

Lemma 1. *For any vector a and nondegenerate matrix A,*

$$a'A^{-1}a = \min_x (x'Ax - 2a'x). \tag{2}$$

Lemma 2. *There exist $\varepsilon > 0$ and $0 < \alpha < 1$ such that, if $\max_{i,j} |t_i - t_j| < \varepsilon$, then*

$$M\left\{\prod_{i=1}^m \dot{\xi}_{t_i}^+ \mid \xi_{t_j} = a,\ j = 1, \dots, m\right\} \geq \alpha\sigma_1 \cdots \sigma_m.$$

Theorem 2. *Suppose:*

1. *The k-th mean-square derivative of the stationary Gaussian process exists.*
2. *In some interval $(0, \delta)$ there exist and are continuous $\rho^{(2k+1)}(t)$ and $\rho^{(2k+2)}(t)$, respectively the $(2k+1)$-st and $(2k+2)$-nd derivatives of the correlation function of ξ_t.*
3. There exist limits

$$\lim_{t\downarrow 0} \frac{\rho^{(2k)}(0) - \rho^{(2k)}(t)}{t} = \rho_+^{(2k+1)}(0) \neq 0,$$

$$\lim_{t\downarrow 0} \frac{\rho_+^{(2k+1)}(0) - \rho^{(2k+1)}(t)}{t} = \rho_+^{(2k+2)}(0).$$

Under these conditions $\alpha_{(m)}(\Delta, a) < \infty$ if and only if $m \leq k^2 + 2k$.

Proof. We find the asymptotics, as $t_i - t_j \to 0$, of the functions

$$\frac{\mu_{i_1} \cdots \mu_{i_l} \sigma_{i_{l+1}} \cdots \sigma_{i_m}}{\sqrt{|R_{11}(t_1, \dots, t_m)|}}. \tag{3}$$

We shall carry out the estimate when all $t_i - t_j \to 0$. In the remaining cases the estimate is analogous. At the same time, for brevity, by lim we mean passage to the limit as $\max |t_i - t_j| \to 0$.

Consider the quantities

$$\zeta_{t_i t_{i+1}}^{(k)} = \frac{\xi_{t_{i+1}}^{(k)} - \xi_{t_i}^{(k)}}{\sqrt{|t_{i+1} - t_i|}}.$$

It is easy to see that

$$\lim M\zeta^{(k)}_{t_i t_{i+1}}\zeta^{(k)}_{t_l t_{l+1}} = \begin{cases} 0, & \text{if } i \neq l, \\ 2\rho_+^{(2k+1)}(0), & \text{if } i = l. \end{cases}$$

Denote, following (1), by $L_{t_1\ldots t_{k+1}}[\xi_t] = [\xi_{t_1}, \ldots, \xi_{t_k}]/k!$, where $[\xi_{t_1}, \ldots, \xi_{t_k}]$ is the k-th divided difference of ξ_t.

The quantities

$$\zeta_{t_1\ldots t_{k+2}} = \frac{L_{t_1\ldots t_{k+1}}[\xi_t] - L_{t_2\ldots t_{k+2}}[\xi_t]}{\sqrt{|t_1 - t_{k+2}|}}$$

possess the same properties as the $\zeta^{(k)}_{t_i t_{i+1}}$ introduced above.

It is known that

$$\mu_i = \frac{|R_{\mu_i}|}{|R_{11}|} = \begin{vmatrix} R_{11} & \frac{\partial\rho(t_i-t_j)}{\partial t_i} \\ \ldots & \ldots \\ a \ \cdots \ a & 0 \end{vmatrix} |R_{11}(t_1, \ldots, t_m)|^{-1},$$

$$\sigma_i^2 = \frac{|R_{\sigma_i}|}{|R_{11}|} = \begin{vmatrix} R_{11} & \frac{\partial\rho(t_i-t_j)}{\partial t_i} \\ \ldots & \ldots \\ \ldots \ \frac{\partial\rho(t_i-t_j)}{\partial t_i} \ \ldots & \frac{\partial^2\rho(0)}{\partial t^2} \end{vmatrix} |R_{11}(t_1, \ldots, t_m)|^{-1}.$$

We perform the following transformations on $|R_{11}|$. At the first step, from each i-th column we subtract the $(i-1)$-st and divide by $t_i - t_{i-1}$, and from each i-th row we subtract the $(i-1)$-st and divide by $t_i - t_{i-1}$.

At the second step we do the same as at the first, without touching the first row and column, and dividing by $t_i - t_{i-2}$, and so on up to the k-th step (see (1)). At the $(k+1)$-st step, from the i-th column ($i = m, \ldots, k+2$) we subtract the $(i-1)$-st and divide by $\sqrt{|t_i - t_{i-k-1}|}$; we do the same with the rows and pass to the limit.

Having carried out analogous transformations on R_{σ_i} and R_{μ_i}, we obtain asymptotic formulas.

$$\frac{1}{2^{m-k-1}} \lim \frac{|R_{11}(t_1, \ldots, t_m)|}{\prod_{l=1,\ldots,k}\prod_{i=l+1,\ldots,m}(t_i - t_{i-l})^2 \prod_{i=k+2,\ldots,m}|t_i - t_{i-k-1}|} = |M\eta\eta'|\,[\rho_+^{(2k+1)}(0)]^{m-k-1},$$

$$\lim \frac{|R_{\sigma_i}(t_1, \ldots, t_m)|}{|R_{11}(t_1, \ldots, t_m)|\prod_{l=1,\ldots,k-1}(t_i - t_{i-l})|t_i - t_{i-k}|} = 2\rho_+^{(2k+1)}(0),$$

$$\lim \frac{|R_{\mu_i}(t_1, \dots, t_m)|}{|R_{11}(t_1, \dots, t_m)| \prod_{l=1,\dots,k-1}(t_i - t_{i-l})\sqrt{|t_i - t_{i-k}|}} = k\rho_+^{(2k+1)}(0),$$

where $\eta = (\xi_{t_0}, \dots, \xi_{t_0}^{(k)})$.

On the basis of these formulas we conclude that, for $m = k^2+2k$, the singularities of the function (3) are integrable, whereas already for $m = k^2+2k+1$ $I(0)$ does not exist.

Corollary. *For Gaussian stationary processes with rational spectral density that have exactly k derivatives in the mean-square sense, the last finite moment of the number of level crossings has order $k^2 + 2k$.*

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Note: Figure translations are in progress. See original paper for figures.