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Abstract

Full Text

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ON THE DIRICHLET PROBLEM FOR MULTIDIMENSIONAL ELLIPTIC SYSTEMS OF SECOND ORDER

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It is known that the Dirichlet problem for a system of differential equations of second order of elliptic type, generally speaking, ceases to be Noetherian. The first examples of such systems of two equations in the plane were constructed by A. V. Bitsadze (^{1}), and the first Noetherian condition was not satisfied—the homogeneous problem had an infinite set of linearly independent solutions. Later B. V. Boyarskii (^{2}) established that, for systems of two second-order equations of elliptic type in spaces of dimensions $n \geqslant 3$, the Dirichlet problem becomes Noetherian. However, for spaces of dimension $n = 4;\ 8$, Yu. T. Antokhin (^{3}) and E. N. Kuzmin (^{4}) constructed examples of systems again with violation of the first Noetherian condition. In the present paper examples of this kind of systems will be constructed in spaces of any even dimension.

Thus, consider in Euclidean space of dimension $2n$ the following two systems of first order (^{5})

$$\mathcal{L}_{\pm}\mathbf{u} = \sum_{\alpha=1}^{n}\left(\mathcal{P}_\alpha \frac{\partial}{\partial x_\alpha} \pm \mathcal{Q}_\alpha \frac{\partial}{\partial y_\alpha}\right)\mathbf{u} = 0. \tag{1}$$

Here x_α, y_α are independent variables, and the matrices $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ are constructed in the form of tensor products:

$$\mathcal{P}_\alpha = 1' \times \cdots \times 1' \times \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \times 1 \times \cdots \times 1,$$

$$\mathcal{Q}_\alpha = 1' \times \cdots \times 1' \times \begin{Vmatrix} 0 & i \\ -i & 0 \end{Vmatrix} \times 1 \times \cdots \times 1, \tag{2}$$

$1' = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}$, $\quad 1 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$, $\quad$ the matrices $P = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$ and $Q = \begin{Vmatrix} 0 & i \\ -i & 0 \end{Vmatrix}$

stand in the α-th place.

The matrices $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ have order 2^n and satisfy the relation

$$\left[\sum_{\alpha=1}^{n} (\mathcal{P}_\alpha \xi_\alpha \pm \mathcal{Q}_\alpha \eta_\alpha)\right]^2 = \left[\sum_{\alpha=1}^{n} (\xi_\alpha^2 + \eta_\alpha^2)\right] \mathcal{E}, \tag{3}$$

where $\mathcal{E}$ is the identity matrix (see (^{6}), p. 363), from which the ellipticity of the systems (1) follows. We construct our elliptic system of second order in the form

$$\Lambda \mathbf{u} = \mathcal{L}_+ \mathcal{L}_- \mathbf{u}. \tag{4}$$

For it the following assertion is true.

Theorem. The homogeneous Dirichlet problem for the system (4) in the ball

$$|z_1|^2 + \cdots + |z_n|^2 \leqslant R^2 \quad (z_\alpha = x_\alpha + i y_\alpha)$$

has as solutions vectors of the form

$$\mathbf{u}_0 = (w_0, 0, \dots, 0), \qquad w_0 = (R^2 - |z_1|^2 - \cdots - |z_n|^2)\, f(z_1^2 + \cdots + z_n^2), \tag{5}$$

where $f(\omega)$ is a function analytic with respect to ω in the image of the ball

$$|z_1|^2 + \cdots + |z_n|^2 \leqslant R^2$$

under the mapping

$$\omega = z_1^2 + \cdots + z_n^2.$$

Proof. To verify that the vector $\mathbf{u}_0$ satisfies the system (4), it is necessary to agree on the order of arrangement of the rows and columns—

rows of the matrices (2). As is known, the tensor product of the matrices $A_1 \times A_2 \times \cdots \times A_n$, $A_\alpha = \{a_{ij}^{(\alpha)}\}$, is the matrix of the form $\{a_{i_1 \dots i_n, j_1 \cdots j_n}\} = \{a_{i_1 j_1}^{(1)} \cdots a_{i_{n_j} n}^{(n)}\}$, composed of all possible products of elements of the factor matrices, where the set of numbers $i_1 \dots i_n$ characterizes a row, and the set $j_1 \dots j_n$ a column. We shall order these sets, and together with them the rows and columns of the matrices $\mathcal{L}_+$ and $\mathcal{L}_-$, in lexicographic order; it is precisely for this ordering rule that our solution $\mathbf{u}_0$ was chosen. Any other ordering of rows (columns) will cause a certain permutation of rows (columns), the same for all our matrices (2); therefore, to obtain a solution it will be necessary to perform the very same permutation of the components of the vector $\mathbf{u}_0$.

Let us find the differential operators of the first column of the matrix Λ; they are obtained by multiplying the matrix $\mathcal{L}_+$ by the first column of the matrix $\mathcal{L}_-$.

In the first column of $\mathcal{L}_-$, in the rows $(1, \dots, 2, \dots, 1)$, $\alpha = 1, \dots, n$, there stand the operators $2\partial/\partial \bar{z}_\alpha = (\partial/\partial x_\alpha + i\partial/\partial y_\alpha)$, while in the remaining rows there are operators identically equal to zero. This is easy to see from the following notation for our operators:

$$\begin{aligned}
\mathcal{L}_+ &= 2\sum_{\alpha=1}^{n} \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix} \times \dots \times \begin{Vmatrix} 0 & \partial/\partial\overline{z}_\alpha \\ \partial/\partial z_\alpha & 0 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \times \dots \times \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \\
\mathcal{L}_- &= 2\sum_{\alpha=1}^{n} \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix} \times \dots \times \begin{Vmatrix} 1 & \partial/\partial z_\alpha \\ \partial/\partial\overline{z}_\alpha & 0 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \times \dots \times \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix},
\end{aligned} \tag{6}$$

$$\partial/\partial z_\alpha = \tfrac{1}{2}(\partial/\partial x_\alpha - i\partial/\partial y_\alpha).$$

The first row of $\mathcal{L}_+$ consists of the operators $2\partial/\partial\overline{z}_\alpha$, standing in the columns $(1, \dots, 2, \dots, 1)$, $\alpha = 1, \dots, n$, and identically equal to zero in the others. Hence the operator obtained by multiplying the first row of $\mathcal{L}_+$ by the first column of $\mathcal{L}_-$ will be

$$4(\partial^2/\partial z_1^2 + \dots + \partial^2/\partial z_n^2). \tag{7}$$

Consider an arbitrary row $(a_1 \dots a_n)$ of the matrix, different from the first. When it is multiplied by the first column of $\mathcal{L}_-$, a nonzero operator can be obtained only in the case when among the a_i there are exactly two twos. Indeed, the operators of the row under consideration are nonzero in the columns $(ca_1, a_2, \dots, a_n), \dots, (a_1, \dots, ca_n)$; $ca_i = 1$ for $a_i = 2$ and $ca_i = 2$ for $a_i = 1$; therefore, if the number of twos among the a_i is not equal to two, the numbers of the columns with operators identically not equal to zero do not coincide with the numbers of those rows of the first column where there also stand operators different from identically zero. Suppose that among the a_i there are two twos,

$$(a_1 \dots a_n) = (1 \dots \underset{(k)}{2} \dots \underset{(l)}{2} \dots 1),$$

then after multiplication one obtains the operator

$$4(-\partial^2/\partial z_l \partial\overline{z}_k + \partial^2/\partial z_k \partial\overline{z}_l), \qquad k, l = 1, \dots, n. \tag{8}$$

This is obtained by a simple computation from (6).

Now it is easy to verify that the function w_0 is a zero of the operators (7) and (8), and consequently the vector $\mathbf{u}_0$ satisfies our system (4).

It is clear that among the solutions of the form (5) of the homogeneous Dirichlet problem there is an infinite set of linearly independent solutions, although they may not exhaust all its solutions. It follows that the Dirichlet problem for our system is not Noetherian.

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Note: Figure translations are in progress. See original paper for figures.

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