RATIONALITY OF MULTIPLE HECKE SERIES OF THE FULL LINEAR GROUP AND SHIMURA' S HYPOTHESIS ON HECKE SERIES OF THE SYMPLECTIC GROUP

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Abstract

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MATHEMATICS

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RATIONALITY OF MULTIPLE HECKE SERIES OF THE FULL LINEAR GROUP AND SHIMURA' S HYPOTHESIS ON HECKE SERIES OF THE SYMPLECTIC GROUP

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1. Let k be a field of p-adic numbers (a finite extension of the field $\mathbf{Q}_p$); D a central division algebra of finite rank over k; $\mathfrak{O}$ a maximal order in D; $\mathfrak{P} = (\Pi)$ the maximal ideal in $\mathfrak{O}$. Put $G = GL(n, D)$, the full linear group of matrices of order n over D;
$I(G) = \{(g_{ij}) \in G,\ g_{ij} \in \mathfrak{O}\}$; $U = \{u \in G,\ u^{\pm 1} \in I(G)\}$. With respect to the natural topology induced by the topology of D, G is a locally compact group; U is an open and compact subgroup of G. Denote by $L(G, U)$ (respectively $L(G, U)_Z$) the algebras of all continuous functions f on G with values in $\mathbf{C}$ (respectively in $\mathbf{Z}$) which, for all $u, u' \in U,\ g \in G$, satisfy the condition

$$f(ugu') = f(g).$$

Multiplication in $L(G, U)$ $(L(G, U)_Z)$ is defined as convolution

$$(f * \varphi)(g) = \int_G f(gh^{-1})\varphi(h)\, dh \qquad (g \in G),$$

where dh is the bi-invariant Haar measure on G; $\int_U dh = 1$.

The algebras $L(G, U)$ and $L(G, U)_Z$, called the Hecke algebras of the group G, play an important role in the theory of automorphic forms and zeta-functions connected with the full linear group. The structure of the Hecke algebras of the group G was determined by Satake ([1]). Let us formulate his result.

Let

$$\Pi_{i,n} = \mathrm{diag}(\underbrace{1, \dots, 1}_{i}, \underbrace{\Pi, \dots, \Pi}_{n-i}), \qquad i = 0, 1, \dots, n-1;$$

for $a \in G$, denote by $\chi(a) = \chi(a)(g)$ the characteristic function of the double class UaU. Then (([1]), §8)

$$L(G, U) = \mathbf{C}[\chi(\Pi_{1,n}), \dots, \chi(\Pi_{n-1,n}), \chi(\Pi_{0,n}^{\pm 1})],$$

$$L(G,U)_Z = \mathbf{Z}[\chi(\Pi_{1,n}), \dots, \chi(\Pi_{n-1,n}), \chi(\Pi_{0,n}^{\pm 1})],$$

and the elements $\chi(\Pi_{i,n})$, $i = 0, 1, \dots, n-1$, are algebraically independent over $\mathbf{C}$.

One of the basic problems in the theory of Hecke algebras is to find an explicit expression for an arbitrary $f \in L(G,U)$ $(L(G,U)_Z)$ in terms of the generators $\chi(\Pi_{i,n})$. The theory of elementary divisors for the group G ([1]) allows one to restrict oneself to considering functions f of the form $\chi(\Pi^{(r_1,\dots,r_n)})$, where

$$\Pi^{(r_1,\dots,r_n)} = \mathrm{diag}(\Pi^{r_1}, \dots, \Pi^{r_n}), \qquad r_i \in \mathbf{Z}, \quad 0 \le r_1 \le r_2 \le \dots \le r_n.$$

It is clear that we can obtain any information of the required kind if we possess sufficiently complete information about the so-called multiple Hecke series of the group G, i.e. the formal series

$$Z_G(t_1, \dots, t_n) = \sum_{0 \le r_1 \le \dots \le r_n} \chi(\Pi^{(r_1,\dots,r_n)}) t_1^{r_1} \cdots t_n^{r_n}. \tag{1}$$

We have proved the following general fact about the series (1):

Theorem 1. *The series (1) is a rational function of $t_1, \dots, t_n$ with coefficients from $L(G,U)_Z$.*

Previously, the rationality of the series (1) was known only in the case when $t_1 = t_2 = \dots = t_n$ ([2]).

2. Theorem 1 admits a reformulation in terms of multiple zeta-functions of the group G. Recall ([1]), §5, that a complex-valued continuous function ω on G is called a zonal spherical function on G (with respect to U) if the following conditions are satisfied:

1) $\omega(ugu') = \omega(g)$ for all $g \in G$, $u, u' \in U$;

2) $\omega(1) = 1$;

3) for every $\varphi \in L(G,U)$, ω is an eigenfunction of the integral operator defined by φ, i.e. $\varphi * \omega = \lambda_\varphi \omega$, where $\lambda_\varphi \in \mathbf{C}$. Denote by $\Omega(G,U)$ the set of all zonal spherical functions on G with respect to U. Put, for $i = 1, \dots, n$ and $g \in G$,

$$V_i(g) = (N\mathfrak{p})^{-r_i},$$

where $UgU = U\Pi^{(r_1,\dots,r_i,\dots,r_n)}U$, $r_1 \le r_2 \le \cdots r_n$, and $N\mathfrak{p}$ is the number of elements of the residue field $\mathfrak{o}/\mathfrak{p}$. We shall call the multiple zeta-function of the group G with character $\omega \in \Omega(G,U)$ the integral

$$\xi_G(s_1, \dots, s_n; \omega) = \int_{I(G)} \omega(g^{-1}) \prod_{i=1}^{n} V_i(g)^{s_i}\, dg, \tag{2}$$

where s_i $(i = 1, \dots, n)$ are complex variables. It is easy to see that the integral (2) converges absolutely in some domain of the form $\min_i \operatorname{Re} s_i > N_\omega$.

Theorem 2. *For any $\omega \in \Omega(G, U)$, the function $\xi_G(s_1, \dots, s_n; \omega)$ extends meromorphically in all variables s_i and is a rational function of the variables $t_i = (N\mathfrak{p})^{-s_i}$ $(i = 1, \dots, n)$.*

Previously, a special case of this theorem was known for $s_1 = s_2 = \dots = s_n$ ([2]).

3. For the field of p-adic numbers k, denote by $\mathfrak{o}$ and $\mathfrak{p} = (\pi)$ the ring of integers of k and the maximal ideal of this ring, respectively. By $|\ |$ we shall denote the normalized norm in k, i.e. $|\xi| = q^{-\operatorname{ord}_\mathfrak{p} \xi}$, where $\xi \in k$, and q is the number of elements of the residue class field $\mathfrak{o}/\mathfrak{p}$. Let $J_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$, where 1_n is the identity matrix of order n. Put

$$S = \operatorname{Sp}(n, k) = \{g \in GL(2n, k);\ {}^{tgJ}_{ng} = r(g) J_n,\ r(g) \in k\}$$

–the symplectic group of genus n over k; $I(S) = S \cap I(GL(2n, k))$, $V = \{v \in S;\ v, v^{-1} \in I(S)\}$. Then, with respect to the natural topology, S is a locally compact group and V is an open and compact subgroup. Analogously to § 1, for the pair (S, V) one can define the Hecke algebras $L(S, V)$ and $L(S, V)_Z$, which are again algebras of polynomials over $\mathbf{C}$ and $\mathbf{Z}$, respectively, with explicitly written generators ([1,3]). Denote by $T_n = T_n(g)$ the characteristic function of the set $X_n = (x \in I(S),\ \operatorname{ord}_\mathfrak{p} r(x) = n)$. Clearly, $T_n \in L(S, V)_Z$. The Hecke series of the group S is the formal series

$$Z_S(t) = \sum_{n=0}^{\infty} T^n_{nt}. \tag{3}$$

Shimura ([3]) conjectured that the series $Z_S(t)$ is a rational function of t. Following ([4]), one can show that Shimura' s conjecture is a consequence of Theorem 2. Thus one has

Theorem 3. *The Hecke series (3) of the group* $\operatorname{Sp}(n, k)$ *is a rational function of t with coefficients in $L(S, V)_Z$.*

Previously, the rationality of the series (3) was known for the cases $n = 1$ ([5]); $n = 2$ ([1,4]); $n = 3$ ([4]). In these cases the series $Z_S(t)$ was computed explicitly.

4. Analogously to § 2, Theorem 3 admits a reformulation in terms of zeta-functions of the group S. Let $\omega \in \Omega(S, V)$ (the definition is analogous to that given in § 2 for $\Omega(G, U)$; see ([1]), §5). The integral

$$\xi_S(s;\omega) = \int_{I(S)} \omega(g^{-1})|r(g)|^s \, dg, \tag{4}$$

where dg is the Haar measure on S; s is a complex variable, is called the zeta-function of the group S with character ω. It is easy to see that the integral (4) converges absolutely in some domain of the form $\operatorname{Re} s > N_\omega$.

Theorem 4. The function $\zeta_S(s;\omega)$ extends meromorphically to the entire s-plane and is a rational function of $t = q^{-s}$.

5. Concerning the proof of Theorems 1-4, we note that Theorems 1, 3, 4 are derived from Theorem 2. To prove Theorem 2, we construct special decompositions for the factor-space $U \setminus GL(n, D)$, reminiscent of the Bruhat decomposition for $GL(n, D)$, which make it possible to parametrize the left classes Ua contained in a given double class UgU $(a, g \in G)$ by points on systems of Grassmann manifolds over residue-class rings modulo powers $\mathfrak{p}$, and we develop a technique for transforming such decompositions into triangular form.

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Note: Figure translations are in progress. See original paper for figures.