# STATISTICAL DYNAMICS OF A TURBULENT INCOMPRESSIBLE FLUID

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## Abstract

## Full Text

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# STATISTICAL DYNAMICS OF A TURBULENT INCOMPRESSIBLE FLUID

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I. **Basic equations. Elimination of pressure.** The Navier–Stokes equations for describing the motion of a viscous incompressible fluid have the form:

$$\partial v_\alpha/\partial t + v_\beta \partial v_\alpha/\partial x_\beta + \partial p/\partial x_\alpha = \nu \Delta v_\alpha, \qquad \partial v_\beta/\partial x_\beta = 0. \tag{1}$$

From these two equations the pressure can be eliminated in the usual way:

$$p(\mathbf{x}) = -\frac{1}{4\pi}\int \frac{\partial v'_\alpha}{\partial x'_\beta}\frac{\partial v'_\beta}{\partial x'_\alpha}\frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} + \psi(\mathbf{x}). \tag{2}$$

The function $\psi(\mathbf{x})$ is harmonic,

$$\Delta\psi(\mathbf{x}) = 0. \tag{3}$$

Although in what follows we shall regard the velocity field as random, we shall show that the function $\psi$ is not a random variable. We shall denote averaging over the ensemble by an overbar and introduce a new random function

$$\psi_1(\mathbf{x}) = \psi(\mathbf{x}) - \overline{\psi(\mathbf{x})}$$

and its two-point second moment

$$\overline{\psi_1(\mathbf{x})\psi_1(\mathbf{x}+\mathbf{r})} = f(\mathbf{x},\mathbf{r}). \tag{4}$$

For fixed $\mathbf{x}$, the function $f(\mathbf{x},\mathbf{r})$ is harmonic. In addition, it satisfies the following physically evident requirements: it must have no singularities and must tend to zero for large $\mathbf{r}$. A harmonic function possessing such properties is equal to zero. Passing in formula (4) to the limit $\mathbf{r} \to 0$, we obtain

$$\overline{\psi_1^2(\mathbf{x})} = 0,$$

which means that the random function $\psi$ coincides exactly with its mean value:

$$\psi(\mathbf{x}) \equiv \overline{\psi(\mathbf{x})}.$$

Thus, in our equations we may regard $\psi(\mathbf{x})$ as not being a random variable.

After integration by parts in formula (2), the equations acquire the form

$$\frac{\partial v_\alpha}{\partial t} + v_\beta \frac{\partial v_\alpha}{\partial x_\beta} - \frac{1}{4\pi} \int v_\beta(\mathbf{x}') v_\gamma(\mathbf{x}') T_{\alpha\beta\gamma}(\mathbf{x}-\mathbf{x}')\, d\mathbf{x}' = -\frac{\partial \overline{\psi}}{\partial x_\alpha} + \nu \Delta v_\alpha,$$

$$\frac{\partial v_\beta}{\partial x_\beta} = 0, \qquad (5)$$

where

$$T_{\alpha\beta\gamma}(\mathbf{x}-\mathbf{x}') = \frac{\partial^3}{\partial x_\alpha \partial x_\beta \partial x_\gamma} \frac{1}{|\mathbf{x}-\mathbf{x}'|}.$$

II. **Chain of equations for distribution functions.** Introduce distribution functions statistically describing the behavior

fluids such that the probability that at the points $\mathbf{x}_1, \dots, \mathbf{x}_n$ at the times $t_1, \dots, t_n$, respectively, the fluid velocities will lie in the intervals $d\mathbf{v}_1, \dots, d\mathbf{v}_n$ is

$$F_n(\mathbf{v}_1, \dots, \mathbf{v}_n; \mathbf{x}_1, \dots, \mathbf{x}_n; \quad t_1, \dots, t_n) d\mathbf{v}_1 \dots d\mathbf{v}_n. \qquad (6)$$

It is clear that the functions $F_n$ give a complete statistical description of the turbulent motion of a fluid. Knowing $F_n$, one can find any mean characteristics, for example the mean dissipated energy

$$\varepsilon(\mathbf{x}_1, t_1) = \nu \, \overline{\frac{\partial v_\alpha(\mathbf{x}_1, t_1)}{\partial x_{1\beta}} \, \frac{\partial v_\alpha(\mathbf{x}_1, t_1)}{\partial x_{1\beta}}} =$$

$$= \nu \int \frac{\partial^2 F_2(\mathbf{v}_1, \mathbf{v}_2; \mathbf{x}_1, \mathbf{x}_2; t_1, t_2)}{\partial x_{1\beta}\, \partial x_{2\beta}} \, v_{1\alpha} v_{2\alpha} \, \delta(\mathbf{x}_1 - \mathbf{x}_2) \delta(t_1 - t_2) \, d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{x}_2 dt_2.$$

Let us derive the chain of equations satisfied by the functions $F_n$. To this end consider an arbitrary function $\varphi(\mathbf{v}_1(\mathbf{x}_1, t'), \dots, \mathbf{v}_n(\mathbf{x}_n, t_n))$ of the velocities at the points $\mathbf{x}_1, t_1, \dots, \mathbf{x}_n, t_n$ of space-time. It is clear that

$$\frac{\partial\varphi}{\partial t_s} = \frac{\partial\varphi}{\partial v_{s\alpha}}\,\frac{\partial v_{s\alpha}(\mathbf{x}_s, t_s)}{\partial t_s},$$

or, using equation (5), we can write

$$\frac{\partial\varphi}{\partial t_s} = \frac{\partial\varphi}{\partial x_{s\alpha}}\left[-v_{s\beta}(\mathbf{x}_s, t_s)\frac{\partial v_{s\alpha}(\mathbf{x}_s, t_s)}{\partial x_{s\beta}} + \right.$$

$$\left. + \frac{1}{4\pi}\int v'_\beta(\mathbf{x}', t_s) v'_\gamma(\mathbf{x}', t_s) T_{\alpha\beta\gamma}(\mathbf{x}_s - \mathbf{x}')\, d\mathbf{x}' + \nu\Delta v_{s\alpha}(\mathbf{x}_s, t_s)\right]. \qquad (7)$$

Take the mean of relation (7). We have

$$\overline{\frac{\partial\varphi}{\partial t_s}} = \int \frac{\partial F_n}{\partial t_s}\,\varphi\, d\mathbf{v}_1 \dots d\mathbf{v}_n,$$

$$\overline{\frac{\partial\varphi}{\partial v_{s\alpha}}\, v_{s\beta}\frac{\partial v_{s\alpha}}{\partial x_{s\beta}}} = \int \frac{\partial F_n}{\partial x_{s\beta}}\, v_{s\beta}\varphi(\mathbf{v}_1, \dots, \mathbf{v}_n)\, d\mathbf{v}_1 \dots d\mathbf{v}_n.$$

The other terms in relation (7) are transformed analogously. Using the fact that $\varphi$ is an arbitrary function, we obtain the chain of equations for the functions $F_n$

$$\frac{\partial F_n}{\partial t_s} = -v_{s\beta}\frac{\partial F_n}{\partial x_{s\beta}} - \frac{1}{4\pi}\int \frac{\partial F_{n+1}}{\partial v_{s\alpha}}\, v_{n+1,\beta} v_{n+1,\gamma} T_{\alpha\beta\gamma}(\mathbf{x}_s - \mathbf{x}_{n+1}) \times$$

$$\times \delta(t_{n+1} - t_s)\, d\mathbf{v}_{n+1} dt_{n+1} d\mathbf{x}_{n+1} - \frac{\partial F_n}{\partial v_{s\alpha}}\,\frac{\partial\overline{\psi}(\mathbf{x}_s, t_s)}{\partial x_{s\alpha}} -$$

$$-\nu\int \delta(\mathbf{x}_s - \mathbf{x}_{n+1})\delta(t_s - t_{n+1})\, d\mathbf{x}_{n+1}\Delta_{n+1}\int \frac{\partial F_{n+1}}{\partial v_{s\alpha}}\, v_{n+1,\alpha}\, d\mathbf{v}_{n+1} dt_{n+1} \;*. \quad (8)$$

If one restricts oneself only to simultaneous functions, then in the right-hand sides of equations (8) one must pass to the limit $t_1 = t_2 = \dots = t$, take into account that

$$\lim_{t_1\to t,\; t_2\to t, \dots,\; t_n\to t}\left[\sum_{s=1}^{n}\frac{\partial F_n}{\partial t_s}\right] = \frac{\partial F_n(\mathbf{v}_1, \dots, \mathbf{v}_n; \mathbf{x}_1, \dots, \mathbf{x}_n; t)}{\partial t}$$

and obtain for the simultaneous correlation functions the chain of equa-

* Equations (8), as we learned while preparing the article for press, were obtained by another method for the

$$\frac{\partial F_n}{\partial t} = -\sum_{k=1}^{n} v_{k\beta}\frac{\partial F_n}{\partial x_{k\beta}} - \frac{1}{4\pi}\sum_{k=1}^{n}\int \frac{\partial F_{n+1}}{\partial v_{k\alpha}}\, v_{n+1,\beta} v_{n+1,\gamma} T_{\alpha\beta\gamma}(\mathbf{x}_k - \mathbf{x}_{n+1})\, d\mathbf{v}_{n+1} d\mathbf{x}_{n+1} -$$

$$-\nu\sum_{k=1}^{n}\int \delta(\mathbf{x}_k - \mathbf{x}_{n+1})\, d\mathbf{x}_{n+1}\Delta_{n+1}\int \frac{\partial F_{n+1}}{\partial v_{k\alpha}}\, v_{n+1,\alpha}\, d\mathbf{v}_{n+1}. \qquad (9)$$

## III. Additional conditions

We present the conditions that the functions $F_n$ must satisfy.

1. **Normalization conditions:**

$$\int F_{n+1}\, d\mathbf{v}_{n+1} = F_n, \qquad \int F_n\, d\mathbf{v}_1 \dots d\mathbf{v}_n = 1.$$

2. **Continuity conditions:**

$$\lim_{\mathbf{x}_{n+1}\to\mathbf{x}_n,\, t_{n+1}\to t_n} F_{n+1} = F_n \delta(\mathbf{v}_{n+1} - \mathbf{v}_n).$$

3. **Symmetry conditions:**

$$F_n(\dots, \mathbf{v}_s, \dots, \mathbf{v}_k, \dots; \dots, \mathbf{x}_s, \dots, \mathbf{x}_k, \dots; \dots, t_s, \dots, t_k, \dots) =$$
$$= F_n(\dots, \mathbf{v}_k, \dots, \mathbf{v}_s, \dots; \dots, \mathbf{x}_k, \dots, \mathbf{x}_s, \dots; \dots, t_k, \dots, t_s, \dots).$$

4. **Incompressibility condition:**

$$\int \frac{\partial F_n}{\partial x_{k\alpha}}\, v_{k\alpha}\, d\mathbf{v}_k = 0.$$

5. **Compatibility conditions:**

$$\frac{\partial F_n}{\partial x_{k\alpha}} = -\frac{\partial}{\partial v_{k\beta}}\int \delta(\mathbf{x}_{n+1} - \mathbf{x}_k)\frac{\partial F_{n+1}}{\partial x_{n+1,\alpha}}\, v_{n+1,\beta}\, d\mathbf{v}_{n+1} d\mathbf{x}_{n+1}\delta(t_{n+1} - t_k)\, dt_{n+1},$$

$$\frac{\partial F_n}{\partial t_k} = -\frac{\partial}{\partial v_{k\beta}}\int \delta(\mathbf{x}_{n+1} - \mathbf{x}_k)\delta(t_{n+1} - t_k)\frac{\partial F_{n+1}}{\partial t_{n+1}}\, v_{n+1,\beta}\, d\mathbf{v}_{n+1} d\mathbf{x}_{n+1} dt_{n+1}.$$

Only when all the conditions are fulfilled can the functions $F_n$ be regarded as functions with independent variables.

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## CITED LITERATURE

1. A. S. Monin, PMM, **31**, no. 6 (1967).

*Note: Figure translations are in progress. See original paper for figures.*