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MATHEMATICS

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Abstract

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MATHEMATICS

M. L. GOLDMAN

ON ESTIMATES OF INTEGRAL NORMS OF EIGENFUNCTIONS OF THE LAPLACE OPERATOR IN CERTAIN DOMAINS

(Presented by Academician A. N. Tikhonov, 26 II 1968)

Let g be an N-dimensional domain ($N \geq 2$), Γ its boundary, and let

$$\{u_i(x)\}, \quad i = 1, 2, \dots, \tag{1}$$

be an orthonormal system of classical eigenfunctions of the Laplace operator in the domain g, corresponding to the system of eigenvalues $\{\lambda_i\}$; $\lambda_{i+1} \geq \lambda_i$, $\lambda_i \to +\infty$. Thus, the problem under consideration is:

$$\{\Delta u + \lambda u = 0 \text{ in } g, \qquad u|_\Gamma = 0. \tag{2}$$

All the results presented here are also valid for the eigenfunctions of the second boundary-value problem.

In considering questions of convergence of Fourier series with respect to the system (1) for functions $f \in L_p(g)$, $1 < p < 2$, almost everywhere or in the metric $L_p(g)$, estimates are needed for the norms of functions of the system (1) in $L_q(g)$, $q > 2$. Estimates of the same kind for $q < 2$ are necessary for studying the question of the c-property of the system (1). It is said that the system (1) possesses the c-property if from the convergence almost everywhere in g of the series $\sum_i a_i u_i(x)$ it follows that $a_i \to 0$ ($i \to \infty$) (see ([2])).

It had previously been assumed that for any domain g admitting the existence of a classical system of eigenfunctions, this system possesses the c-property and has bounded norms in $L_q(g)$, $q > 2$.

In this paper it is shown that, in the general case, one cannot expect positive results in this direction. Namely, examples are given of domains with arbitrarily smooth boundaries for which the systems (1) do not possess the indicated properties (Theorem 1). In Theorem 2, the existence is shown for the first time of a domain in almost all points of which there is no boundedness, uniform in i, of the eigenfunctions $\{u_i(x)\}$.

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Let g_1 be an N-dimensional domain obtained as the metric product of the disk K_2 by the $(N-2)$-dimensional cube $P_{N-2} = \{0 \le x_j \le \pi\}$, $j = 1, 2, \ldots, N-2$.

Let K_N be an N-dimensional ball. The radii of K_2 and K_N $(N \ge 2)$ are assumed equal to 1.

Theorem 1. There exists a monotonically increasing sequence of indices $n_k \to \infty$ such that for the eigenfunctions (1) in the domain g_1 the estimate

$$c_1 \lambda_{n_k}^{(q-2)/6q} \le \|u_{n_k}(x)\|_{L_q(g_1)} \le c_2 \lambda_{n_k}^{(q-2)/6q}, \tag{3}$$

holds, where $q > 2$ is arbitrary and the constants $c_2 \ge c_1 > 0$.

Remark. For the eigenfunctions of the N-dimensional ball K_N, for $q > 2$ the left-hand side of estimate (3), i.e., the lower estimate, is valid, and for $q < 2$ the right-hand side of (3) (the upper estimate) is valid.

The eigenfunctions K_N can be written explicitly. Let (r, Ω) be a spherical coordinate system. Then

$$u_{n,m}(r, \Omega) =$$

$$= J_{n+(N-2)/2}(r\mu^m_{n+(N-2)/2}) P_n(\Omega) \,/\, r^{(N-2)/2} \left\{ \int_0^1 J^2_{n+(N-2)/2}(r\mu^m_{n+(N-2)/2}) r\, dr \right\}^{1/2}, \tag{4}$$

where $P_n(\Omega)$ is the normalized angular part; $\mu^m_{n+(N-2)/2}$ is the m-th zero of the Bessel function of index $\nu = n + (N-2)/2$ (the zeros are numbered by m in increasing order). We shall consider the subsequence of eigenfunctions with $m = 1$ $(n \to \infty)$. For the proof of the theorem we shall need the following lemma.

Lemma 1. *Let*

$$I^{(q)} = \int_0^1 \left| \frac{J_{n+(N-2)/2}(r\mu^1_{n+(N-2)/2})}{r^{(N-2)/2}} \right|^q r^{N-1}\, dr, \tag{5}$$

where $q \geqslant 1$; $N \geqslant 2$ is fixed; $\nu = n + (N-2)/2$, $n = 1, 2, \ldots$. Then, as $n \to \infty$ $(\nu \to \infty)$, the estimate

$$c_1'/(\mu^1_\nu)^{(q+2)/3} \leqslant I^{(q)} \leqslant c_2'/(\mu^1_\nu)^{(q+2)/3}, \tag{6}$$

holds, where $c_2' \geqslant c_1' > 0$.

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The proof of Lemma 1 is based on the use of V. A. Fock' s asymptotic formulas (see ([3])) for Bessel functions $J_\nu(x)$ of large positive order.

For $0 < x \leqslant \nu$,

$$J_\nu(\nu\ \mathrm{sech}\,\alpha) \simeq \frac{1}{\pi}\sqrt{\alpha\ \mathrm{cth}\,\alpha - 1}\,K_{1/3}[\nu(\alpha - \mathrm{th}\,\alpha)], \tag{7}$$

where $0 \leqslant \alpha < \infty$, $K_{1/3}[x]$ is the Macdonald function, and $\mathrm{sech}\,\alpha = 1/\,\mathrm{ch}\,\alpha$. For $x \geqslant \nu$,

$$J_\nu(\nu \sec\beta) \simeq \sqrt{1 - \beta\,\mathrm{ctg}\,\beta}\left\{\cos\frac{\pi}{6}\cdot J_{1/3}[\nu(\mathrm{tg}\,\beta - \beta)] - \right.$$

$$\left. - \sin\frac{\pi}{6}\,Y_{1/3}[\nu(\mathrm{tg}\,\beta - \beta)]\right\}, \tag{8}$$

where $0 \leqslant \beta < \pi/2$, and $J_{1/3}$ and $Y_{1/3}$ are the Bessel and Neumann functions, respectively. (For the properties of Bessel functions see ([1]).) Formulas (7) and (8) hold uniformly for $x \leqslant \nu$ and $x \geqslant \nu$, respectively.

Make in $I^{(q)}$ (see (5)) the change of variables $x = r\mu_\nu^1$. Then

$$I^{(q)} = (\mu_\nu^1)^{q(N-2)/2-N}\int_0^{\mu_\nu{}^1}\frac{|J_\nu(x)|^q\,dx}{x^{q(N-2)/2-(N-1)}}. \tag{9}$$

It is known that $\mu_\nu^1 > \nu$.

Splitting

$$\int_0^{\mu_\nu{}^1} = \int_0^\nu + \int_\nu^{\mu_\nu{}^1} = I_1 + I_2, \tag{10}$$

we make in I_1 the substitution $x = \nu\ \mathrm{sech}\,\alpha$, and in I_2 the substitution $x = \nu\sec\beta$, and obtain

$$I_1 = \nu^{N-q(N-2)/2}\int_0^\infty \frac{|J_\nu(\nu\ \mathrm{sech}\,\alpha)|^q\ \mathrm{sh}\,\alpha\,d\alpha}{(\mathrm{ch}\,\alpha)^{N-1-q(N-2)/2}\,\mathrm{ch}^2\,\alpha}, \tag{11}$$

$$I_2 = \nu^{N-q(N-2)/2}\int_0^{\beta_0(\nu)} \frac{|J_\nu(\nu\sec\beta)|^q\sin\beta\,d\beta}{(\cos\beta)^{N-1-q(N-2)/2}\cos^2\beta}, \tag{12}$$

where $\sec\beta_0(\nu) = \mu_\nu^1/\nu$.

Since it is known that

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$$\mu_\nu^1 = \nu + c\nu^{1/3} + O\left(\nu^{-1/3}\right) \tag{13}$$

(see (1)), we have $\sec \beta_0(\nu) \approx 1 + c/\nu^{2/3}$, and $\beta_0(\nu) \approx c/\nu^{1/3}$.

The estimate of the integral (11) is carried out with the aid of formula (7). In doing so one must take into account that $K_{1/3}(x) \approx c'/x^{1/3}$ for $0 < x \leq x_0$, $K_{1/3}(x) \approx \sqrt{\pi/2x}\, e^{-x}$ for $x > x_0$, and split the region of integration correspondingly into two:

$$1) \quad 0 \leq \alpha < \alpha_0(\nu), \tag{14}$$

where $\nu[\alpha_0(\nu) - \operatorname{th}\alpha_0(\nu)] = x_0$, and

$$2) \quad \alpha_0(\nu) \leq \alpha < \infty. \tag{15}$$

In the region (14) one can obtain a two-sided estimate, while in the region (15) an upper estimate is sufficient. After some calculations, for I_1 one obtains the estimate

$$c_1''\nu^{N-q(N-2)/2}/\nu^{(q+2)/3} \leq I_1 \leq c_2''\nu^{N-q(N-2)/2}/\nu^{(q+2)/3}. \tag{16}$$

To estimate I_2 (see (12)) one must use (8) and take into account that $|Y_{1/3}(x)| \approx c'/x^{1/3}$ for $0 < x \leq x_0$; $J_{1/3}(x) \leq cx^{1/3}$ for $0 \leq x \leq x_0$. Then for I_2 we obtain

$$I_2 \leq c''\nu^{N-q(N-2)/2}/\nu^{(q+2)/2}. \tag{17}$$

From (16) and (17), taking into account (13) and (9)–(10), we easily prove the principal estimate (6). Lemma 1 is proved.

For the proof of Theorem 1, let us write out the eigenfunctions of the domain g_1

$$u_{(i)} = \frac{J_n(r\mu_n^m)\begin{matrix}\sin n\varphi\\ \cos n\varphi\end{matrix}}{c\left[\int_0^1 J_n^2(r\mu_n^m)\, r\, dr\right]^{1/2}} \sin k_1 x_1 \ldots \sin k_{N-2} x_{N-2}, \tag{18}$$

where c is the normalizing constant, and (i) is the set of indices: $(i) = \{n, m, k_1 \ldots k_{N-2}\}$.

Put $m = k_1 = \ldots = k_{N-2} = 1$, i.e. consider the subsequence (i_k). Then

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$$\lambda_{(i)} = (\mu_n^m)^2 + \sum_{j=1}^{N-2} k_j^2, \qquad \lambda_{(i_k)} = (\mu_n^1)^2 + (N-2). \tag{19}$$

By Lemma 1 ($N = 2$) we have the estimate:

$$\frac{c_1'}{(\mu_n^1)^{2/3}} \left[\int_0^1 J_n^2(r\mu_n^1)\, r\, dr \right]^{1/2} \leq \frac{c_2'}{(\mu_n^1)^{2/3}}. \tag{20}$$

The estimate is also valid

$$0 < c_1'' \leq \left[\int_0^{2\pi} |\sin n\varphi|^q \, d\varphi \right]^{1/q} \leq c_2'', \qquad n = 1, 2, ... \tag{21}$$

Then, estimating in (18) the normalizing factor by (20), and applying Lemma 1 and estimate (21) to the numerator, we immediately arrive at the assertion of the theorem (3).

The fact noted in the remark follows from Lemma 1, formula (4), and the inequalities

$$\|P_n(\Omega)\|_{L_q(\Omega)} \geq c > 0 \quad \text{for } q > 2, \tag{22}$$

$$\|P_n(\Omega)\|_{L_q(\Omega)} \leq c' < \infty \quad \text{for } q < 2, \tag{23}$$

where (22) and (23) follow from Hölder' s inequality for integrals and from the fact that $\|P_n(\Omega)\|_{L_2(\Omega)} = 1$.

Corollary 1. *The eigenfunctions of the Laplace operator in the domain g_1 and in the ball K_N do not possess the c-property ($N \geqslant 2$).*

Indeed, it is known (see ([2])) that for a system to possess the c-property it is necessary and sufficient that $\lim\limits_n \|u_n(x)\|_{L_1(g)} > 0$, which is not fulfilled in view of estimate (3) and the remark to Theorem 1 for $q = 1$.

Corollary 2. *There exist functions $f(x) \in L_p$ (for any $1 < p < 2$) whose Fourier coefficients with respect to the system of eigenfunctions of the domain g_1 or of the ball K_N are unbounded. The set of such functions has, in L_p, a complement of first category.*

Let us note that the Fourier coefficients

$$c_n(f) = \int f(x) u_n(x)\, dx$$

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form a sequence of linear bounded functionals in L_p with norm $\|c_n\| = \|u_n\|_{L_q}$, where $1/p + 1/q = 1$. But by Theorem 1 ($q > 2$) we have

$$\overline{\lim_n} \|u_n\|_{L_q} = \infty.$$

It remains to refer to the Banach–Steinhaus theorem (see, for example, ([2]), § 5).

Corollary 3. *Let $\{m_n\}$ be any sequence of indices for which*

$$\mu_n^{m_n}/n \xrightarrow[n\to\infty]{} 1.$$

Then the eigenfunctions (18) of the domain g_1, for any $k_1, \dots, k_{N-2}$ and $m = m_n$, converge uniformly to zero as $n \to \infty$ in every strictly interior closed subdomain $g' \Subset g$.

The proof of this fact is obtained by applying formula (7), taking into account the estimate of the normalizing factor:

$$\int_0^1 J_n^2(r\mu_n^{m_n})\, r\, dr = \frac{1}{(\mu_n^{m_n})^2} \int_0^{\mu_n^{m_n}} |J_n^2(x)|\, x\, dx \geqslant \frac{1}{(\mu_n^{m_n})^2} \int_0^{\mu_n^1} J_n^2(x)\, x\, dx \geqslant \frac{c}{n^{4/3}},$$

which follows, for example, from (16) ($N = 2,\ q = 2$).

The question remains open as to what conditions on a domain ensure the existence of the c-property for the eigenfunctions of the Laplace operator.

Theorem 2. *For almost every point $x \in g_1$ one can choose a subsequence of indices (i_k), depending on x, so that at this point the estimate*

$$|u_{(i_k)}(x)| \geqslant c\lambda_{(i_k)}^{1/12}, \tag{24}$$

holds, where $c > 0$ is a constant (depending on x).

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.

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