THE STRUCTURE OF THE SET OF SOLUTIONS OF DISSIPATIVE EQUATIONS

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Abstract

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MATHEMATICS

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THE STRUCTURE OF THE SET OF SOLUTIONS OF DISSIPATIVE EQUATIONS

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In the present paper we study the disposition of integral curves of periodic dissipative equations with unbounded operators in a Banach space. It is established that every solution of a dissipative equation asymptotically approaches the Λ-center of the equation—a compact set of solutions $u(t)$, defined and bounded on $(-\infty, \infty)$, possessing the property that, under a shift by the period along the trajectories of these solutions, the set of initial values is mapped onto itself. This fact generalizes the known theorems on dissipative systems of ordinary differential equations ($^{1-3}$). In connection with the investigation of dissipative systems, new fixed-point principles are established. The notion of the χ-center of a dissipative equation is introduced and, with the aid of somewhat modified results of G. F. Hilmi (^{9,10}), it is shown that every solution of this equation "almost all the time" lies in an arbitrarily small neighborhood of the χ-center (this result, apparently, had not been observed even for dissipative systems of ordinary differential equations).

The equations studied in the article include broad classes of boundary-value problems for evolutionary partial differential equations. The simplest example is the problem

$$\partial u/\partial t = Lu + \Psi(t,x,u), \qquad u(t,x) = 0 \quad \text{for } x \in \Gamma, \tag{1}$$

where x ranges over the closed bounded domain $\overline{\Omega}$ of a finite-dimensional space with smooth boundary Γ; L is an elliptic negative definite differential operator of second order; the function $\Psi(t,x,u)$ is sufficiently smooth and ω-periodic in t; problem (1) may be regarded as a dissipative equation in the space C of functions continuous on Ω, provided $\Psi(t,x,u)$ is bounded.

1. A partially ordered (see (^{4})) set R will be called regularly partially ordered if in it there is distinguished a fixed element z_0 and if every such sequence $x_n \in R$ that

$$z_0 \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots, \tag{2}$$

has an exact lower bound $\inf x_n$. A regularly partially ordered set will be called completely regularly partially ordered if every transfinite sequence x_α such that $z_0 \geq x_\alpha \geq x_\beta$ for $\alpha < \beta$ has an exact lower bound. A partially ordered set is called directed if for any two elements $x, y \in R$ there exists an element z such that $x \leq z,\ y \leq z$.

An operator T acting in a partially ordered set R is called monotone if $x \leq y$ implies $Tx \leq Ty$. A monotone operator T acting in a regularly partially ordered set R will be called dissipative if for every $x \in R$ one can find an $n = n(x)$ such that $T^n x \leq z_0$.

Theorem 1. *Let R be a completely regularly partially ordered directed set. Let T be a monotone dissipative operator. Then the set Π of fixed points of the operator T is nonempty, and in the set Π there is a greatest element w.*

We shall call a monotone operator directionally continuous if, for every sequence x_n satisfying condition (2), the equality

$$\inf Tx_n = T(\inf x_n)$$

holds. If, for example, R is a metric space and $\rho(x_k, \inf x_n) \to 0$ as $k \to \infty$ for every sequence x_n satisfying condition (2), then directional continuity follows from the ordinary continuity of the operator.

Theorem 2. *Let R be a regular partially ordered directed set. Let T be a monotone dissipative directionally continuous operator. Then the assertions of Theorem 1 are valid.*

We note that, in proving the existence of at least one fixed point under the conditions of Theorems 1 and 2, the assumption that the operator is dissipative may be replaced by the less restrictive assumption that there exists one such element $x \leq z_0$ for which $T^n x \leq x$ and $T^m x \leq x$, where m and n are relatively prime integers.

2. A regularly partially ordered directed set will be called a regularly partially ordered space if convergence of certain sequences is defined in R, and moreover: a) sequences x_n satisfying condition (2) converge, and $\lim x_n = \inf x_n$; b) if $x_n \leq y_n \leq z_n$, where x_n and z_n satisfy condition (2), and if $\lim x_n = \lim z_n$, then the sequence y_n converges and $\lim y_n = \lim x_n = \lim z_n$; c) if the sequences $x_n^{(i)}$, $i = 1, 2, \dots, s$, have a common limit y, then the sequence z_k, where $z_{sn+i} = x_n^{(i)}$, also converges and $\lim z_k = y$.

Let R denote the system of closed bounded subsets of a Banach space E with a distinguished compact set z_0. We introduce an order by inclusion: $x \leq y$ if the set x is a subset of the set y. Obviously, R is a completely regular partially ordered directed set, since the infimum of a sequence of compact closed sets nested in one another coincides with their intersection. If the Hausdorff metric is defined in R, then R becomes a regularly partially ordered space.

Theorem 3. *Let R be a regularly partially ordered space. Let T satisfy the conditions of Theorem 2. Finally, let $x_0 \geq w$, where w is the greatest element in the set Π of fixed points of the operator T. Then the sequence $x_n = T^n x_0$ $(n = 1, 2, ...)$ converges to the point w.*

3. Consider, in a Banach space E, the ordinary differential equation

$$du/dt = f(t, u). \tag{3}$$

We shall assume that the operator $f(t, u)$ is ω-periodic in t, i.e.,

$$f(t + \omega, u) = f(t, u).$$

Next, we shall suppose that for each $u_0 \in E$ there is uniquely determined a solution $\varphi(t; t_0, u_0)$ of equation (3), satisfying the initial condition $\varphi(t_0; t_0, u_0) = u_0$ and defined for all $t \geq t_0$. Recall that a solution satisfies the equation for $t > t_0$ and is continuous for $t \geq t_0$. We shall assume that the solution depends continuously on the initial data t_0, u_0. By $U(t)$ we denote the shift operator by time t along trajectories of equation (3); this operator is defined (see ([3],[7])) by the equality

$$U(t)u_0 = \varphi(t; 0, u_0).$$

We shall say that equation (3) is dissipative if the operator $U(\omega)$ is completely continuous and if there exists a $d > 0$ such that every solution $\varphi(t; t_0, u_0)$ of equation (3) satisfies the condition

$$\overline{\lim_{t\to\infty}} \|\varphi(t; t_0, u_0)\| < d. \tag{4}$$

Conditions for the complete continuity of the operator $U(\omega)$ have been established for broad classes of equations in a Banach space (see, for example, ([7]),

Ch. 5). Conditions (4) are in many cases also verified without difficulty. Let us emphasize that the operator $f(t, u)$ is not assumed to be continuous.

As follows from one theorem of Browder ([8]), each dissipative equation has at least one ω-periodic solution.

Let $\mathfrak{M}$ be some set of functions defined on $(-\infty, \infty)$ with values in E. By $M(t^*)$ we shall denote the set of values at $t = t^*$ of all functions from $\mathfrak{M}$. A set $\mathfrak{M}$, closed with respect to uniform convergence on the whole axis, of solutions of equation (3) bounded on $(-\infty, \infty)$ will be called its Λ-center if $U(\omega)M(0) = M(0)$ and if every solution $\varphi(t; t_0, u_0)$ of equation (3) satisfies the condition

$$\lim_{t\to\infty} \rho[\varphi(t; t_0, u_0),\, M(t)] = 0. \tag{5}$$

Theorem 4. *Every dissipative equation (3) has a Λ-center, compact in the sense of uniform convergence on every finite interval of variation of time.*

Theorem 4 admits various generalizations. For example, for the existence of a compact (in the above sense) Λ-center it is sufficient that condition (4) be satisfied and that some power of the operator $U(\omega)$ be completely continuous.

4. Let $\mathfrak{M}$ be the Λ-center of the dissipative equation (3). Denote by D the set of points $\{\tau, u\}$, where $0 \leq \tau \leq \omega$, $u \in M(\tau)$, in which the points $\{0, u\}$ and $\{\omega, u\}$ are identified. The differential equation (3) induces on D a semigroup of transformations

$$V_t\{\tau, u\} = \{\tau_1, \varphi(t+\tau, \tau, u)\},$$

where $\tau_1 = \tau + t (\operatorname{mod} \omega)$ $(t, \tau \geq 0)$. In general the semigroup V_t cannot be extended to a group, since we do not assume that the solutions of equation (3) are uniquely determined by their initial value as t decreases.

The set D with the semigroup V_t will be called a semidynamical system. To such systems the basic facts (^{9,10}) of the theory of minimal and quasi-minimal sets are transferred without difficulty. In particular, for each point $p = \{t, u\} \in D$ one can define the minimal center of attraction $S(p)$ of Hilmy (^{9}). Put

$$\mathfrak{M}_0 = \bigcup_{p \in D} S_p.$$

If $\mathfrak{M}_0$ is regarded as a set of trajectories of equation (3), then it possesses the property $M_0(t+\omega) = M_0(t)$. Therefore $\mathfrak{M}_0$ may be regarded as a set of solutions of equation (3) defined for $-\infty < t < \infty$. We shall call the set $\mathfrak{M}_0$ the χ-center of equation (3).

Theorem 5. *Let $\mathfrak{M}_0$ be the χ-center of equation (3). Then for any solution $\varphi(t; t_0, u_0)$ and for any $\varepsilon > 0$ the equality*

$$\lim_{t_1 \to \infty} (t_2 - t_0)^{-1} \operatorname{mes}\{t : t_0 \leq t \leq t_1, \rho[\varphi(t; t_0, u_0),\, M_0(t)] > \varepsilon\} = 0.$$

5. Theorems 4 and 5 extend in a natural way to the case when the shift operator $U(\omega)$ does not have the property of complete continuity (and is even not defined on the whole space E), if one can choose a linear operator A acting in E (as a rule, unbounded) such that, for some $a > 0$, the operator $U_1(\omega) = A^{aU}(\omega)A^{-a}$ admits an extension to all of E to a completely continuous operator. This remark (see ($^{5-7}$)) substantially broadens the range of applications.

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Note: Figure translations are in progress. See original paper for figures.

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