ON THE THEORY OF CANONICAL DIFFERENTIAL OPERATORS IN HILBERT SPACE

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Abstract

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MATHEMATICS

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ON THE THEORY OF CANONICAL DIFFERENTIAL OPERATORS IN HILBERT SPACE

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1. Let $\mathfrak{R}$ be a separable Hilbert space, and let $\mathfrak{B}$ be the ring of continuous operators in $\mathfrak{R}$. For any interval E of the real axis and any subspace $\mathfrak{R}' \subset \mathfrak{R}$, by $L_2(E, \mathfrak{R}')$ we shall denote the space of measurable functions $f(x)$, $x \in E$, with values in $\mathfrak{R}'$, satisfying the condition

$$\int_E \|f(x)\|^2_{\mathfrak{R}}\, dx < \infty.$$

By $L(E, \mathfrak{B})$ we shall denote the space of functions with values in $\mathfrak{B}$ that are Bochner-integrable on E.

Choose in $\mathfrak{B}$ some operator J with the properties: $J^* = -J$, $J^2 = -I$, where I is the identity operator. For J, evidently, there is the spectral decomposition $J = iP_+ - iP_-$, where $P_\pm$ are orthogonal projectors, $P_\pm = \frac{1}{2}(J \pm iI)$, which decompose $\mathfrak{R}$ into two subspaces $\mathfrak{R}_+$ and $\mathfrak{R}_-$, $\mathfrak{R}_\pm = P_\pm \mathfrak{R}$. We assume that $\dim \mathfrak{R}_+ = \dim \mathfrak{R}_-$. By virtue of this assumption one can construct an isometric operator Z mapping $\mathfrak{R}_-$ onto $\mathfrak{R}_+$, and form the subspace $\mathfrak{R}_0 = \{h : h = e + Ze,\ e \in \mathfrak{R}_-\}$. From the definition of $\mathfrak{R}_0$ it follows that $(Jf, g) = 0$ for all $f, g \in \mathfrak{R}_0$ and that the subspace $\mathfrak{R}_0$ admits no extensions in $\mathfrak{R}$ with preservation of this property. In what follows the operators J and Z are regarded as fixed.

The purpose of the present note is to study the self-adjoint differential operator A in $L_2((0, \infty); \mathfrak{R})$, formally defined by the expression

$$(Af)(x) = -J\, df(x)/dx + V(x)f(x) \tag{1}$$

and by the boundary condition

$$f(0) \in \mathfrak{R}_0, \tag{2}$$

where $V(x)$ is a function from $L(0, \infty; \mathfrak{B})$ with self-adjoint values. We also impose on $V(x)$ the additional condition

$$JV(x) = -V(x)J, \tag{3}$$

which in fact does not impair the generality of the consideration, since one can always achieve its fulfillment by passing from A to an isomorphic operator $G^* AG$, where $G = G(x)$ is a function whose values are unitary operators in $\mathfrak{R}$, defined by the equalities

$$-J\, dG/dx = -\{P_+ V(x) P_+ + P_- V(x) P_-\} G, \qquad G(0) = I.$$

Differential operators of the form (1) under the condition $\dim \mathfrak{R} < \infty$, i.e., in those cases to which the radial equations of quantum theory for relativistic particles and, in particular, the radial Dirac equation lead, have already been considered in works ([1–3]), where, under the assumption of local summability of the function $V(x)$, the direct problem of scattering theory was studied and a solution was found to the inverse problems of reconstructing $V(x)$ from the spectrum and from scattering data. (Some particulars and details concerning ([2]) can be found in the article ([4]).) Inverse problems for the radial Dirac equation were also studied in recent works ([5,6]).

Below, without assuming that $\dim \mathfrak{R} < \infty$, but requiring that $V(x) \in L(0, \infty; \mathfrak{B})$, we give for the operator A an analogue of the eigenfunction expansion theorem and a solution of the direct problem of scattering theory, and, in the case when the function $V(x)$ is finite, we establish the applicability to the operator A of the variant of scattering theory proposed by P. Lax and R. Phillips ([7]). We also give some results concerning operators of the form (1) with a non-self-adjoint function $V(x)$.

2. Consider the differential equation

$$-J\, dF(x,\lambda)/dx + V(x)F(x,\lambda) - \lambda F(x,\lambda) = 0, \tag{4}$$

where λ is a real parameter. A function $F(x, \lambda)$ with values in $\mathfrak{B}$ will be called a solution of equation (4) if, for $F(x, \lambda)$, equality (4) holds in the norm of the space $\mathfrak{B}$.

Denote by $\mathcal{E}(x, \lambda)$ the solution of equation (4) satisfying the condition $\mathcal{E}(0, \lambda) = I$. Note that there exists only one such function $\mathcal{E}(x, \lambda)$, and every function $F(x, \lambda)$ satisfying (4) and the condition $F(0, \lambda) = C$, where $C \in \mathfrak{B}$, is obtained from the relation $F(x, \lambda) = \mathcal{E}(x, \lambda)C$.

For $\mathcal{E}(x, \lambda)$ the equalities

$$\mathcal{E}^*(x,\lambda) J \mathcal{E}(x,\lambda) = \mathcal{E}(x,\lambda) J \mathcal{E}^*(x,\lambda) = J \tag{5}$$

and the representation

$$\mathcal{E}(x,\lambda)=\exp(J\lambda x)+\int_0^x \exp[J\lambda(x-s)]\,\Gamma(x,s)\,ds, \tag{6}$$

hold, where $\Gamma(x,s)$ $(0<s<x,\ 0<x<\infty)$ belongs to the space of functions with values in $\mathfrak{B}$, measurable in the norm of $\mathfrak{B}$, and such that

$$\sup_x \int_0^x \|\Gamma(x,s)\|\,ds<\infty.$$

For the determination of $\Gamma(x,s)$ one uses the "integral" equation

$$\Gamma(x,s)=-JV(s)-J\int_s^x V(t)\Gamma(t,t-s)\,dt, \tag{7}$$

which has a unique solution in the indicated space of functions, provided $V(s)\in L(0,x;\mathfrak{B})$.

As $x\to\infty$, the sequence $\{\exp(-J\lambda x)\mathcal{E}(x,\lambda)\}$ converges uniformly in λ, and

$$G(\lambda)=\lim_{x\to\infty}\exp(-J\lambda x)\mathcal{E}(x,\lambda)=I+\int_0^\infty \exp(-J\lambda s)\Gamma(s)\,ds,$$

where $\Gamma(s)$ is the limit in $L(0,\infty;\mathfrak{B})$ of the convergent sequence of functions $\{\Gamma_x(s)\}$, and $\Gamma_x(s)=\Gamma(x,s)$ for $x>s$ and $\Gamma_x(s)=0$ for $x<s$.

The estimates

$$\exp\left\{-\int_0^\infty \|V(s)\|\,ds\right\}\cdot I\le\{G^*(\lambda)G(\lambda)\}^{1/2}\le\exp\left\{\int_0^\infty \|V(s)\|\,ds\right\}\cdot I \tag{8}$$

are valid.

3. Let $\Phi(x,\lambda)$ $(\operatorname{Im}\lambda=0)$ be the solution of equation (4) satisfying the condition $\Phi(0,\lambda)=P_0$, where P_0 is the orthogonal projector onto the subspace $\mathfrak{R}_0$, i.e. $\Phi(x,\lambda)=\mathcal{E}(x,\lambda)P_0$. By virtue of (8), the self-adjoint operator $P_0G^*(\lambda)G(\lambda)P_0$ is continuously invertible in $\mathfrak{R}_0$, and for the corresponding inverse operator $\Delta(\lambda)$ the estimates

$$\exp\left\{-2\int_0^\infty \|V(s)\|\,ds\right\}P_0\le\Delta(\lambda)\le\exp\left\{2\int_0^\infty \|V(s)\|\,ds\right\}P_0. \tag{9}$$

hold.

Denote by $L_2^\Delta(-\infty,\infty;\mathfrak{R}_0)$ the space of measurable functions with values in $\mathfrak{R}_0$ in which the norm is given by the formula

$$\|f\|_\Delta = \left\{\frac{1}{\pi}\int_{-\infty}^{\infty}(\Delta(\lambda)f(\lambda),f(\lambda))\,d\lambda\right\}^{1/2}.$$

Inequality (9) shows that the spaces $L_2(-\infty,\infty;\mathfrak{N}_0)$ and $L_2^\Delta(-\infty,\infty;\mathfrak{N}_0)$ consist of the same functions.

Theorem 1. For any function $f(x)\in L_2(0,\infty;\mathfrak{N})$ the representation

$$f(x) = \text{l. i. m.}_{N\to\infty}\frac{1}{\pi}\int_{-N}^{N}\Phi(x,\lambda)\Delta(\lambda)\tilde{f}(\lambda)\,d\lambda, \tag{10}$$

is valid, where $\tilde{f}(\lambda)\in L_2^\Delta(-\infty,\infty;\mathfrak{N}_0)$ and is found by the inversion formula

$$\tilde{f}(\lambda) = \text{l. i. m.}_{R\to\infty}\int_0^R\Phi^*(x,\lambda)f(x)\,dx. \tag{11}$$

There is an analogue of Parseval' s equality

$$\int_0^\infty\|f(x)\|^2\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty}d\lambda\,(\Delta(\lambda)\tilde{f}(\lambda),\tilde{f}(\lambda)). \tag{12}$$

The assertion of Theorem 1 follows from the relations and estimates given above if, in addition, one assumes that $V(x)=0$ for $x>a>0$ and $\sup_x\|V(x)\|<\infty$. In the general case it is enough to note that the set of functions $V(x)$ from $L(0,\infty;\mathfrak{B})$ with self-adjoint values for which Theorem 1 is true is closed in $L(0,\infty;\mathfrak{B})$.

Under the mapping (10) of the space $L_2^\Delta(-\infty,\infty;\mathfrak{N}_0)$ onto $L_2(0,\infty;\mathfrak{N})$, the operator of multiplication by the variable λ in $L_2^\Delta(-\infty,\infty;\mathfrak{N}_0)$ generates a self-adjoint operator A, defined on the set dense in $L_2(0,\infty;\mathfrak{N})$ which is the image of the set

$$\left\{f(\lambda);\ \frac{1}{\pi}\int_{-\infty}^{\infty}\lambda^2(\Delta(\lambda)f(\lambda),f(\lambda))\,d\lambda\right\}.$$

By the self-adjoint operator A generated by the differential expression (1) and condition (2), one should understand precisely this operator. The operator A has a homogeneous Lebesgue spectrum of multiplicity $\dim\mathfrak{N}_0=\dim\mathfrak{N}_\pm$.

4. Denote by A_0 the differential operator of the type under consideration corresponding to the case $V(x)\equiv 0$.

Theorem 2. There exist and are unitary wave operators

$$W_\pm(A,A_0) = s\text{-}\lim_{t\to\pm\infty}e^{iAt}e^{-iA_0t}. \tag{13}$$

The action of the operators $W_\pm(A, A_0)$ is determined by the formulas

$$(W_\pm(A, A_0)f)(x) = \mathrm{l.\,i.\,m.}_{N\to\infty} \int_{-N}^{N} \Phi(x,\lambda)\Delta(\lambda)A_\mp(\lambda)\tilde{f}(\lambda)\,d\lambda,$$

$$\tilde{f}(\lambda) = \mathrm{l.\,i.\,m.}_{R\to\infty} P_0 \int_0^R \exp\{-J\lambda x\} f(x)\,dx, \tag{14}$$

where

$$A_\mp(\lambda) = 2P_0 G^*(\lambda) P_\mp P_0.$$

From the formal properties of wave operators and Theorem 2 it follows that

Corollary. The operators A and A_0 are isomorphic, and

$$W_\pm(A, A_0)E^0_\lambda W^*_\pm(A, A_0) = E_\lambda,$$

where E_λ and E^0_λ are the resolutions of the identity of the operators A and A_0.

Using the wave operators, we construct the scattering operator

$$S = (A, A_0) = W^*_+(A, A_0)W_-(A, A_0).$$

On the basis of (14), for any function $f(x) \in L_2(0,\infty;\mathfrak{N})$ we have

$$(S(A, A_0)f)(x) = \mathrm{l.\,i.\,m.}_{N\to\infty} \frac{1}{\pi} P_0 \int_{-N}^{N} \exp(J\lambda x) A_-^{-1}(\lambda)A_+(\lambda)\tilde{f}(\lambda)\,d\lambda,$$

$$\tilde{f}(\lambda) = \mathrm{l.\,i.\,m.}_{R\to\infty} P_0 \int_0^R \exp(-J\lambda x) f(x)\,dx. \tag{15}$$

It is seen from (15) that the scattering suboperator (the Geisenberg scattering matrix) is computed by the formula

$$S(\lambda) = A_-(\lambda)A_+^{-1}(\lambda). \tag{16}$$

Let us note that the functions $A_+(\lambda)$ and $A_-(\lambda)$ are boundary values of functions analytic respectively in the upper and lower half-planes and factorize the spectral density $\Delta(\lambda)$ in the form

$$\Delta(\lambda) = \left(A_\pm(\lambda)A^*_\pm(\lambda)\right)^{-1}. \tag{17}$$

5. Suppose that $V(x) = 0$ for $x > a > 0$, and denote by $\mathfrak{D}_\pm$ the subspaces $L_2(0,\infty;\mathfrak{N}_\pm)$ of the space $L_2(0,\infty;\mathfrak{N})$. With respect to the group of unitary operators $U_t = \exp(-iAt)$, the subspaces $\mathfrak{D}_\pm$ have the following properties:

$$
\begin{aligned}
&1)\quad U_{\pm t}\mathfrak{D}_\pm \subset \mathfrak{D}_\pm, \quad t>0; \qquad 2)\quad \bigcap_t U_t\mathfrak{D}_\pm = \{0\};\\
&3)\quad \bigcup_t U_t\mathfrak{D}_\pm = L_2(0,\infty;\mathfrak{N}); \quad 4)\quad \mathfrak{D}_+ \perp \mathfrak{D}_-.
\end{aligned} \tag{18}
$$

In view of (18), the group U_t is isomorphic to the groups of left shifts in the spaces $L_2(-\infty,\infty;\mathfrak{N}_\pm)$. This isomorphism is established by isometric operators $F_\pm$, whose action on any function $f(x)\in L_2(0,\infty;\mathfrak{N})$ is defined by the formulas

$$
(F_\pm f)(s) = P_\pm (U_s f)(x)\big|_{x=a}. \tag{19}
$$

Here

$$
F_\pm\mathfrak{D}_\pm = \{g(s) : g(s)\in L_2(-\infty,\infty;\mathfrak{N}_\pm),\ g(\pm s)=0,\ s>0\}.
$$

To groups of unitary operators U_t for which there exist subspaces $\mathfrak{D}_\pm$ with properties (18), the scheme of scattering theory developed in recent works of P. Lax and R. Phillips for the study of wave equations [7] is applicable. The group U_t, generated by the differential operator (1) with a finite function $V(x)$, is thus another object to which this scheme is applicable.

6. We now abandon the assumption that the values of the function $V(x)$ are self-adjoint. Then, if equality (3) holds and

$$
\int_0^\infty \|V(x)\|\,dx < \ln 2, \tag{20}
$$

then, as before, the functions $\Phi(x,\lambda)$ ($\operatorname{Im}\lambda = 0$) form a complete system in $L_2(0,\infty;\mathfrak{N})$, and the non-self-adjoint operator A, defined by expression (1) and condition (2), admits an exact definition analogous to that given above for the self-adjoint case.

Theorem 3. *If the function $V(x)$ satisfies conditions (3) and (20), then the group of operators $T_t = \exp(-iAt)$, $-\infty < t < \infty$, is uniformly bounded; there exist continuously invertible wave operators*

$$
W_\pm(A,A_0) = s\text{-}\lim_{t\to\pm\infty} T_{-t}U_t^0,
$$

where $U_t^0 = \exp(-iA_0 t)$ is the group of unitary operators generated by the operator A_0 considered; the groups of operators T_t and U_t^0 are similar, and

$$
T_t = W_\pm(A,A_0)U_t^0 W_\pm^{-1}(A,A_0).
$$

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Note: Figure translations are in progress. See original paper for figures.