ON A CLASS OF PROBLEMS WITH AN UNKNOWN BOUNDARY FOR THE HEAT-CONDUCTION EQUATION

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Abstract

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ON A CLASS OF PROBLEMS WITH AN UNKNOWN BOUNDARY FOR THE HEAT-CONDUCTION EQUATION

(Presented by Academician A. Yu. Ishlinskii, 15 IV 1967)

The present work arose from consideration of the problem of the impact of a viscoplastic rod against a rigid obstacle, which was posed in the work of G. I. Barenblatt and A. Yu. Ishlinskii (^{1}) and which served as the basic model for isolating the class of problems with an unknown boundary investigated below.

The formulation of the problems is as follows. It is required to find, on some interval $[0, T]$, a continuous function $h(t)$, $h(0) = 0$, $h(t) > 0$ for $t \in (0, T)$, and a bounded solution $u(t, x)$ of the heat-conduction equation

$$u_t = u_{xx} \tag{1}$$

in the domain

$$\Omega = \{(t, x) : 0 < x < h(t),\ 0 < t \leq T\},$$

continuous in $\overline{\Omega} \setminus (0, 0)$ together with the derivative $u_x(t, x)$, and satisfying the conditions

$$u(t, 0) = f(t), \quad u_x(t, h(t)) = 0, \tag{2}$$

$$u(t, h(t)) = A(h) \equiv g_h(t), \tag{3}$$

where $f(t)$ is a function continuous for $t \geq 0$, and A is some operator with values in the space of continuously differentiable functions. The unknown boundary $x = h(t)$ is sought in some subset of continuous functions, which we shall call the set of admissible boundaries; the choice of this set, which may be connected with physical considerations or with the concrete form of the operator A, is an essential point in the formulation of the problem.

We agree in what follows not to emphasize the dependence of the function g in (3) on $h(t)$, except in those cases where this is necessary.

The problem of the impact of a viscoplastic rod is a particular case of problem (1)–(3) when

$$f(t) \equiv 0, \qquad A(h) = 1 - s\int_0^t \frac{d\tau}{1-h(\tau)}, \qquad s = \text{const} > 0, \tag{4}$$

where $u(t,x)$ here is the velocity of the point of the rod with coordinate x at the moment of time t; the set of admissible boundaries in the present case is the collection of functions $h(t)$, continuous for $t \geq 0$, satisfying the conditions: $h(0) = 0$, $0 < h(t) < 1$ for $0 < t < T$ (the length of the rod and its velocity at the moment of impact in dimensionless variables are equal to 1).

The unknown functions in solutions of the problems under consideration, as a rule, have the following peculiarities: the solution of the heat-conduction equation is discontinuous at the point $(0,0)$, the unknown boundary is nonmonotone, and its derivatives are unbounded.

In the present work a uniqueness theorem for the solution of these problems is established, and functional equations for the unknown boundary are given.

are equivalent to the original problem) and some properties of the solutions are discussed.

Theorem 1. Let the functions $h(t)$, $u(t,x)$ be a solution of problem (1)–(3). Then

$$\begin{aligned} u(t,x) = g(0)\Phi\left(\frac{1}{2\sqrt{t}}\right) + \frac{1}{2\sqrt{\pi}}\int_0^t \frac{x}{(t-\tau)^{3/2}}\exp\left(-\frac{x^2}{4(t-\tau)}\right) f(\tau)\,d\tau \\ + \frac{1}{2}\int_0^t \left[\Phi\left(\frac{x+h(\tau)}{2\sqrt{t-\tau}}\right) + \Phi\left(\frac{x-h(\tau)}{2\sqrt{t-\tau}}\right)\right] g'(\tau)\,d\tau, \end{aligned} \tag{5}$$

where

$$\Phi(\sigma) = (2/\sqrt{\pi})\int_0^\sigma \exp(-\sigma^2)\,d\sigma.$$

The following corollaries follow from Theorem 1: 1) the function

$$v(t,x) \equiv u(t,x) - g(0)\Phi\left(\frac{x}{2\sqrt{t}}\right) - f(0)\left(1 - \Phi\left(\frac{x}{2\sqrt{t}}\right)\right),$$

extended by zero at $(0,0)$, is continuous in $\overline{\Omega}$, and $v(t,x) \to 0$ as $t \to 0$; 2) the function $u(t,x)$ is discontinuous at $(0,0)$ if and only if

$$f(0) \neq g(0); \tag{6}$$

3) if condition (6) is satisfied, then

$$\lim_{t\to 0} \frac{h(t)}{\sqrt{t}} = +\infty. \tag{7}$$

Theorem 2. Let the operator A on the set of admissible boundaries possess the following properties:

1) $A(h)|_{t=0} = g(0)$ does not depend on the choice of $h(t)$;
2) if $h_1(t_0) > h_2(t_0)$, then there exists a point $t^* < t_0$ such that $h_1(t^*) = h_2(t^*)$, and for $g_{h_i} = A(h_i)$ $(i = 1, 2)$ the inequality

$$g_{h_1}(t^*) - g_{h_1}(t_0) \geq g_{h_2}(t^*) - g_{h_2}(t_0)$$

holds.

Let the function $f(t)$ be continuously differentiable for $t \geq 0$ and

$$f(0) \leq g(0), \qquad f'(t) \leq 0\, * . \tag{8}$$

Then the solution of problem (1)–(3) is unique.

If $A(h) \equiv F(h(t))$, where $F(\sigma)$ is a smooth decreasing function, then property 2) of the operator A is satisfied for any t_0 with $t^* = 0$. Another example of such an operator is given by the problem of impact of a rod (see (4)). We shall present the main points in the proof of Theorem 2 for the case of this problem.

First of all, from Theorem 1, using the strengthened maximum principle (^{2}), we obtain that for any solution $h(t)$, $u(t,x)$ of problem (1)–(4) the inequalities

$$u_x(t,x) \geq 0 \quad \text{in } \overline{\Omega} \setminus (0,0), \tag{9}$$

$$u_x(t,x) > 0 \quad \text{in } \Omega. \tag{10}$$

hold.

Suppose that there exist two solutions: $h_1(t)$, $u_1(t,x)$ on $[0,T_1]$ and $h_2(t)$, $u_2(t,x)$ on $[0,T_2]$. Let $T = \min[T_1,T_2]$, $m(t) = \min[(h_1(t), h_2(t))]$, $D = \{(t,x) : 0 < x < m(t), 0 < t \leq T\}$. Consider in the domain $\overline{D}$ the function $w(t,x) = u_1(t,x) - u_2(t,x)$, $w(0,0) = 0$, which satisfies equation (1) in D. By virtue of the first corollary of Theorem 1, this function is continuous in $\overline{D}$, and $w(t,x) \to 0$ as $t \to 0$.

Let P be the point of maximum of the function $w(t,x)$ in $\overline{D}$, with $w(P) > 0$. It is clear that $P = (t_0, m(t_0))$, $t_0 \in (0,T]$, and that $h_1(t_0) > h_2(t_0)$, since otherwise the known property $w_x(P) > 0$ would contradict the consequence of condition (2) for u_1 and inequality (9) for u_2: $w_x(P) = u_{1x}(P) - u_{2x}(P) = -u_{2x}(P) \leqslant 0$. Denote by t^* the upper face of those $t < t_0$ for which $h_1(t) = h_2(t)$; evidently $h_1(t^*) = h_2(t^*)$ and $h_1(t) > h_2(t)$ for $t^* < t < t_0$. Hence, using conditions (2)–(4) and inequality (10) for u_1, we find

$$w(P) = u_1(t_0, m(t_0)) - \left(1 - s\int_0^{t_0} \frac{d\tau}{1 - h_2(\tau)}\right) < s\int_0^{t_0} \left(\frac{1}{1-h_2} - \frac{1}{1-h_1}\right) d\tau <$$

$$< s\int_0^{t^*} \left(\frac{1}{1-h_2} - \frac{1}{1-h_1}\right) d\tau = w(t^*, m(t^*)),$$

which contradicts the choice of the point P. Thus $w(P) \leqslant 0$, and everywhere in $\overline{D}$ $u_1 \leqslant u_2$; by symmetry (in the proof) of the functions u_1 and u_2 the reverse inequality also holds. Thus $u_1 = u_2$ in $\overline{D}$. Let us show that $h_1(t) \equiv h_2(t)$ on $[0,T]$. Indeed, if there existed a point $\theta \in (0,T]$ such that, for example, $h_1(\theta) < h_2(\theta)$, then by condition (2) $u_{1x}(\theta, h_1(\theta)) = 0$, while by inequality (10) $u_{1x}(\theta, h_1(\theta)) = u_{2x}(\theta, h_1(\theta)) > 0$–a contradiction.

From Theorem 1 there follow functional equations for the unknown boundary. For example,

$$g(0)\left[1 - \Phi\left(\frac{h(t)}{2\sqrt{t}}\right)\right] = \frac{1}{2\sqrt{\pi}}\int_0^t \frac{h(t)}{(t-\tau)^{3/2}} \exp\left(-\frac{h^2(t)}{4(t-\tau)}\right) f(\tau)\, d\tau -$$

$$-\frac{1}{2}\int_0^t \left[2 - \Phi\left(\frac{h(t)+h(\tau)}{2\sqrt{t-\tau}}\right) - \Phi\left(\frac{h(t)-h(\tau)}{2\sqrt{t-\tau}}\right)\right] g'(\tau)\, d\tau. \qquad (11)$$

Two other equations can be obtained from the conditions $u_x(t, h(t)) = 0$ and $u_t|_{x=h(t)} = du/dt = g'(t)$. Under certain natural assumptions, the equivalence of these equations to problem (1)–(3) is established.

From equation (11) there follows the following necessary condition for the solvability of problem (1)–(3):

$$\operatorname{sign}[g(0) - f(0)] = -\operatorname{sign} g'(0);$$

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* The case $f(0) \geq g(0)$, $f'(t) \geq 0$ is reduced to (8) by multiplying the data of problem (1)–(3) by -1.

moreover, from (11) one can obtain the asymptotics of $h(t)$ as $t \to 0$:

$$h(t) \sim 2\sqrt{t \ln 1/t}. \tag{12}$$

From conditions (3), (4) and the maximum principle it follows that, for the interval $(0, T)$ of existence of a solution of the rod-impact problem, the estimate $T \leqslant 1/s$ is valid, with $h(T) = 0$. It is not difficult to see that for the unknown boundary in this problem the following lower estimate is valid on the whole interval of existence of the solution: $h(t) \geqslant 2\sqrt{t}\,\omega(t)$, where $\omega(t)$ is the unique solution of the differential equation $\omega' = -s/\pi e^{\omega^2}/(1 - 2\sqrt{t}\,\omega)$, satisfying the condition $\omega(+0) = +\infty$; it can be shown that the function $\omega(t)$ vanishes at some $t = T_0$ and, consequently, $T \geqslant T_0$. It has been established that $\sqrt{T - t} = o(h(t))$ as $t \to T$ (condition (7) for $h(t)$ in the present case was obtained in [3]).

The results of the work extend to the case of problems with an initial condition and to certain equations with variable coefficients.

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Note: Figure translations are in progress. See original paper for figures.

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