ON STRONGLY ASYMMETRIC SEQUENCES GENERATED BY A FINITE NUMBER OF SYMBOLS

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Abstract

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ON STRONGLY ASYMMETRIC SEQUENCES GENERATED BY A FINITE NUMBER OF SYMBOLS

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We consider sequences composed of a finite number of symbols. i consecutive symbols of such a sequence form a segment of length i ($i = 1, 2, 3, \dots$). Two consecutive segments are called equal if they consist of identical symbols and each symbol occurs in both segments the same number of times. The result of the present note is the construction of an infinite sequence, composed of a finite number of symbols, in which no two consecutive segments are equal. The question of the possibility of constructing such a sequence using a minimal number of symbols was posed as a problem by P. Erdős in paper ([1]) and arose, apparently, as a natural generalization of some previously known problems ([2–4]).

1. Consider the sequence of 5 symbols

$$abcdeacbcebdedaedcba. \tag{*}$$

We agree that the i-section of the sequence $(*)$ leaves on the left i terms of the sequence ($i = 1, 2, \dots, 19$). Introduce the column

$$L_i = \begin{pmatrix} a_a \\ a_b \\ a_c \\ a_d \\ a_e \end{pmatrix},$$

where a_x is the number of occurrences of the symbol x in the left part of the i-section of the sequence $(*)$. Similarly, the column P_i of the occurrences of symbols in the right part of the i-section of the sequence $(*)$ is defined.

Lemma 1. The sequence $(*)$ has the following properties:

a) it contains no consecutive equal segments;

b) L_i and P_i for $i \in [2, 18]$ (i.e., i takes natural values from the indicated interval) contain at least two odd numbers, and for $i = 1, 19$ one odd number a_a;

c) the column $(P_i - L_i)$ for $i \in [2, 18]$ contains at least two numbers equal in absolute value to 2, and for $i = 1, 19$ one number $|a_a| = 2$ (this property follows easily from b);

d) the column $(P_i + L_j)$ contains only even numbers if and only if either $i = j$, or $i + j = 20$ and $i \notin [7, 13]$;

e) the column $[L_{(10+i) \bmod 20} - L_i]$ for $i \neq 5, 15$ contains at least two odd numbers, and for $i = 5, 15$ all numbers are equal in absolute value to 2.

2. Let $\{x_0, x_1, \dots, x_{n-1}\}$ be a set of symbols. Form n subsets

$$M_i = \{x_i, x_{(i+1) \bmod n}, x_{(i+3) \bmod n}, x_{(i+7) \bmod n}, x_{(i+12) \bmod n}\} \qquad (i = 0, 1, \dots, n-1).$$

Lemma 2. If $n \geq 25$, then $M_i \cap M_j$ for $i \neq j$ contains no more than one symbol.

Indeed, if M_i and M_j contain at least two common symbols, then

$$(i + p) \bmod n = (j + q) \bmod n, \qquad (i + u) \bmod n = (j + v) \bmod n, \qquad (1)$$

where p, q, u, v take values from the set $\{0, 1, 3, 7, 12\}$ and $p \neq u$, $q \neq v$, $p \neq q$, $u \neq v$.

From (1) it follows that

$$q + u \equiv p + v \pmod{n}. \qquad (2)$$

A direct check shows that, under the restrictions indicated above, identity (2) is impossible.

3. Let the symbols $\{x_0, x_1, \dots, x_{n-1}\}$ $(n \geq 25)$ be cyclically ordered

$$x_0 \to x_1 \to \dots \to x_{n-1} \to x_0. \qquad (3)$$

From the symbols $x_0, x_1, x_3, x_7, x_{12}$, as from a, b, c, d, e, respectively, form the sequence $(*)$. We shall call this sequence the zero block. The i-block ($i = 1, 2, 3, \dots, n-1$) is formed from the $(i-1)$-blocks by a cyclic replacement of symbols according to (3). Since each of the n blocks is obtained by some replacement of the symbols in $(*)$, each block has properties a)–e) of Lemma 1. It is easy to see that the i-block is composed of the symbols of the set M_i, and therefore distinct blocks have no more than one common symbol.

Denote by S_i the column of occurrences of symbols in the i-block,

$$S_i = \left\| \begin{matrix} \alpha_{i0} \\ \alpha_{i1} \\ \vdots \\ \alpha_{i(n-1)} \end{matrix} \right\|,$$

where α_{ij} is the number of occurrences of the j-th symbol in the i-block. By the construction of the i-block we have

$$\alpha_{ii} = \alpha_{i,i+1} = \alpha_{i,i+7} = \alpha_{i,i+12} = 4,$$

and all the other numbers in the column S_i are equal to zero. Addition and multiplication of the columns S_i by numbers are defined in the usual way.

Lemma 3. *The collection of columns* $\{S_i\}$ $(i = 0, 1, \dots, n-1)$ *forms a basis of the* n*-dimensional vector space over the rational field.*

It is enough to show that the determinant D of the matrix $\|\alpha_{ij}\|$ of order n, whose columns are the columns $\{S_i\}$, is nonzero. By the construction of the blocks the matrix $\|\alpha_{ij}\|$ is circulant (5), and therefore the value of its determinant is equal to

$$D = f(\varepsilon_0) f(\varepsilon_1) \cdots f(\varepsilon_{n-1}),$$

where

$$f(x) = 4 + 4x + 4x^3 + 4x^7 + 4x^{12},$$

and ε_k are all the values of $\sqrt[n]{1}$ $(n \geq 25)$. But it is not hard to show that the polynomials $f(x)$ and $x^{25} - 1$ are relatively prime, i.e. $f(\varepsilon_k) \neq 0$ for $k = 0, \dots, 24$.

4. Let N_0 be an arbitrary sequence without consecutive equal segments, composed of the symbols $\{x_0, x_1, \dots, x_{n-1}\}$. Replace in N_0 each symbol x_i by the i-block. We shall call the resulting sequence the 1-iteration and denote it by N_1. Obviously, N_1 consists of the symbols $x_0, x_1, \dots, x_{n-1}$. In general, the m-iteration N_m $(m = 2, 3, \dots)$ is formed from the $(m-1)$-iteration by replacing in it the symbols x_i by the i-blocks. The iterative process described makes it possible to construct arbitrarily long sequences composed of a finite number of symbols. If one takes the symbol x_0 as N_0, then N_1 is the zero block, and by induction it is easy to prove that every sequence N_m is an initial segment of N_{m+1}, i.e. in this case the iterative process defines the construction of an infinite sequence.

Theorem. *The sequence* N_m $(m = 1, 2, 3, \dots)$ *contains no consecutive equal segments.*

It is proved that the presence in N_m of consecutive equal segments either entails their presence in N_{m-1}, or contradicts Lemmas 1, 2, 3. The assumed equality

Fig. 1

Figure 1: Fig. 1

in N_m of consecutive segments is represented schematically (see Fig. 1). The rectangles represent the blocks composing

sequence N_m. The vertical lines determine the boundaries of consecutive equal segments. The letters α and i at a line mean that this line makes an i-section of an α-block (i.e., to the left of the line there are i symbols of the α-block). As in 1, for every α-block the columns L_i^α and P_i^α of occurrences of symbols in the left and right parts of the i-section are introduced. $L_i^\alpha + P_i^\alpha = S_\alpha$. By construction of the blocks, L_i^α and P_i^α differ from L_i and P_i for $(*)$ only by zeros indicating the absence in the α-block of the corresponding symbols from $\{x_0, x_1, \dots, x_{n-1}\}$.

Fig. 1

The column $F = (P_i^\alpha + L_r^\beta - P_r^\beta - L_j^\gamma)$ characterizes the asymmetry of the symbols created by the sections i, r, j of the blocks α, β, γ. We consider all possible cases of equality of segments in N_m

$$\text{I.} \quad (i = j = 0) \quad \begin{cases} r = 0; & (11) \\ r = 10. & (12) \end{cases}$$

$$\text{II.} \quad (i = 0,\ j \neq 0) \quad (i \neq 0,\ j = 0 \text{ is proved analogously}).$$

$$\text{III.} \quad (i \neq 0,\ j \neq 0) \quad \begin{cases} (\alpha \neq \gamma) \begin{cases} i \in [2,18]\ (j \in [2,18]); & (\text{III}\,1) \\ i = 1, 19; \quad j = 1, 19; & (\text{III}\,2) \end{cases} \\ (\alpha = \gamma) \begin{cases} i = j; & (\text{III}\,3) \\ i + j = 20; \quad i \notin [7,14]; & (\text{III}\,4) \\ i, j \text{ do not satisfy (III 3) and (III 4).} & (\text{III}\,5) \end{cases} \end{cases}$$

(11) F is the zero column. Remove identical blocks from both consecutive segments. If, after removal of the blocks, nothing remains in the segments, this means that each block entered both segments the same number of times, but then the corresponding consecutive segments in N_{m-1} are equal, which is impossible. Suppose that after deletion, in the left segment there remain i_1-, i_2-, ..., i_p-blocks with multiplicities of occurrence $a_1, a_2, \dots, a_p$, respectively, and in the right segment j_1-, j_2-, ..., j_q-blocks with multiplicities of occurrence $b_1, b_2, \dots, b_q$. The condition of equality of the segments is then written in the form

$$\sum_{l=1}^{q} b_l S_{j_l} - \sum_{l=1}^{p} a_l S_{i_l} = F, \qquad (**)$$

where a_l and b_l are natural numbers.

But in our case F is the zero column, and therefore $(**)$ means a linear dependence of the columns $\{S_i\}$, which is impossible by Lemma 3.

(12) $L_{10}^{\beta} - P_{10}^{\beta}$, by property c) of Lemma 1, contains numbers whose absolute values are equal to 2. Under this condition the equality $(**)$ is impossible, since the columns $\{S_i\}$ consist only of zeros and fours, and consequently any linear combination of them with integer coefficients is a column of numbers divisible by four.

(II) Let us note that from the condition of equality of consecutive segments it necessarily follows that the column $(P_i^{\alpha} + L_j^{\alpha})$ consists only of even numbers. In our case, by property b) of Lemma 1, this is not satisfied.

(III 1) P_i^{α}, by property b) of Lemma 1, contains at least two odd numbers. But, by Lemma 2, the α- and γ-blocks $(\alpha \neq \gamma)$ have no more than one common symbol.

one common symbol, and therefore, when adding Π_i^{α} to Π_j^{γ}, at least one of the odd numbers of the column Π_i^{α} will be added to a zero of the column Π_j^{α}, i.e. $(\Pi_i^{\alpha} + \Pi_j^{\gamma})$ contains an odd number.

(III 2) As in (III 1), $\Pi_i^{\alpha} + \Pi_j^{\gamma}$, in view of $\alpha \neq \gamma$ and $i = 1, 19,\ j = 1, 19$, contains two odd numbers.

(III 3) Four subcases are possible:

$$(r = i) = \begin{cases} \longrightarrow \alpha = \beta; & \text{(III 3a)} \\ \longrightarrow \alpha \neq \beta; & \text{(III 3b)} \end{cases}$$

$$r = (10 + i) \bmod 20 = \begin{cases} \longrightarrow \alpha = \beta; & \text{(III 3c)} \\ \longrightarrow \alpha \neq \beta. & \text{(III 3d)} \end{cases}$$

We shall prove only one of them.

(III 3c) The case $i = 10$ is proved by (I 2). Let $i \neq 10$.

$$F = \Pi_i^{\alpha} + \Pi_{(10+i) \bmod 20}^{\alpha} - \Pi_{(10+i) \bmod 20}^{\alpha} - \Pi_i^{\alpha} = 2(\Pi_{(10+i) \bmod 20}^{\alpha} - \Pi_i^{\alpha}).$$

If $i \neq 5, 15$, then, by property e) of Lemma 1, F contains either the numbers ± 2, or ± 6, i.e. equality $(**)$ is impossible. If, however, $i = 5, 15$, then, by property e) of Lemma 1, $F = \pm S_{\alpha}$. In view of the uniqueness of the expansion of F with respect to the basis $\{S_i\}$, in condition $(**)$ the coefficient of S_{α} is equal to ± 1, and all the others are equal to zero, i.e. after deleting identical nonintersecting blocks from both segments, only one α-block remains in the right or left segment.

It is not hard to see that then in N_m there exist two consecutive equal segments for which $i = j = r = 0$, which is impossible by (I 1).

(III 4) Can be reduced to case (I 1) or (I 2).

(III 5) By property d) of Lemma 1, the column $\Pi_i^\alpha + \Pi_j^\alpha$ contains odd numbers.

The cases enumerated exhaust all possible dispositions of consecutive segments in N_m.

5. **Remark.** The number of symbols necessary for the indicated construction of the sequence may be reduced. It is possible that $n = 5$. It is easy to show that $n \neq 3$.

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Note: Figure translations are in progress. See original paper for figures.