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Abstract

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MATHEMATICS

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INVESTIGATION OF SMALL SOLUTIONS OF A CLASS OF NONLINEAR OPERATOR EQUATIONS

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1. Let A be a nonlinear operator acting in a Banach space E . We shall assume that the operator A admits the representation

$$Ax = Bx + Cx + Dx, \quad (1)$$

where B is a linear operator, C is a quadratic operator, and D contains terms of third order of smallness:

$$\|Dx\| \leq p\|x\|^3.$$

We shall assume that the spectrum of the linear operator B consists of the point 1 and of some set F in the complex plane lying in the circle $|z| \leq r_0 < 1$. We shall suppose that the root subspace E_0 corresponding to the eigenvalue 1 (see, for example, ⁽¹⁾) is two-dimensional and consists only of eigenvectors. Then, as is known, the space E can be represented as the direct sum of the subspace E_0 and a certain subspace E^0 invariant for B . The spectrum of the restriction B^0 of the operator B to the subspace E^0 coincides with the set F (see ⁽²⁾). Every $x \in E$ can be represented in the form $x = x_0 + x^0$ ($x_0 \in E_0$, $x^0 \in E^0$); by P and Q we denote the projection operators defined by the equalities $Px = x_0$, $Qx = x^0$.

Let $r_0 < r_1 < 1$. Since the spectrum of the restriction B^0 of the operator B to the subspace E^0 lies in the circle $|z| \leq r_0$, one can introduce in E^0 such an equivalent norm $\|x\|^0$ that

$$\|Bx\|^0 \leq r_1 \|x\|^0 \quad (x \in E^0).$$

By $\|x\|_0$ we denote some Euclidean norm in the plane E_0 . Then on the entire space E the norm

$$\|x\|_* = \|Px\| + \|Qx\|^0 \quad (x \in E).$$

will be defined. This norm is equivalent to the original one.

Throughout the article it is assumed that the two-dimensional vector field PCx on the two-dimensional subspace E_0 is nondegenerate; this means that $PCx \neq 0$ if $x \in E_0$ and $x \neq 0$. If in E_0 one chooses some coordinate system, then the components of the field PCx will be quadratic forms in the coordinates of the point x .

We shall be interested in the problem of the arrangement in the space E of the set of small solutions of the equation

$$Ax = (1 + \mu)x \quad (2)$$

for small real μ . This problem has been studied in detail by various methods for the case when 1 is a simple eigenvalue of the operator B . The results set forth below are obtained by methods of the theory of cones; closely related assertions can also be obtained by analytic methods.

2. An important role in our constructions will be played by the eigendirections of the operator PC . If, in some coordinate system $\{\xi, \eta\}$, the field PCx is given by the equality

$$PCx = \{a_1\xi^2 + 2b_1\xi\eta + c_1\eta^2; a_2\xi^2 + 2b_2\xi\eta + c_2\eta^2\}, \quad (3)$$

then the angular coefficient k of the eigendirections is determined from the cubic equation

$$c_1k^3 + (2b_1 - c_2)k^2 + (a_1 - 2b_2)k + a_2 = 0. \quad (4)$$

(if $c_1 = 0$, then the η -axis is an eigendirection of the operator PC). The fundamental case is that in which all three roots of equation (4) are simple. This case is considered in the paper.

Denote by $\mathfrak{N}(\mu_0, \rho_0)$ the set of nonzero solutions of equation (2) for $|\mu| \leq \mu_0$, lying in the ball $\|x\| \leq \rho_0$.

Theorem 1. *Let A be completely continuous. Then, for any ρ_0, μ_0 , the set $\mathfrak{N}(\mu_0, \rho_0)$ is nonempty and forms a continuous branch in a neighborhood of zero (in the sense that on the boundary of each sufficiently small neighborhood of zero there is at least one point of the set $\mathfrak{N}(\mu_0, \rho_0)$).*

Each eigendirection of the operator PC determines two eigenrays, one of which consists of eigenvectors corresponding to positive eigenvalues, and the other of eigenvectors corresponding to negative eigenvalues.

Denote by Π the union of all eigenrays l_i of the operator PC . If equation (4) has one simple real root, then Π consists of 2 rays; and if equation (4) has 3 real roots, then Π consists of 6 rays.

Theorem 2. *The set $\mathfrak{N}(\mu_0, \rho_0)$ is tangent at zero to Π in the sense that*

$$\lim_{x \rightarrow 0, x \in \mathfrak{N}(\mu_0, \rho_0)} \frac{\rho(x, \Pi)}{\|x\|} = 0. \quad (5)$$

Equality (5) can be sharpened: the estimate $\rho(x, \Pi) = O(\|x\|^{3/2})$ is valid. From the general theorems on bifurcation points (see, for example, ^(3,4)) there follows another important relation: $\mu(x) \rightarrow 0$ as $x \rightarrow 0$ ($x \in \mathfrak{N}(\mu_0, \rho_0)$), where $\mu(x)$ is the value of the parameter μ for which x is a solution of equation (2).

3. Let K_i be a certain cone in the sense of M. T. Krein (see ^(5,6)), inside which lies the eigenray l_i . It follows from Theorem 2 that the small solutions of equation (2) from the set $\mathfrak{N}(\mu_0, \rho_0)$ lie in the union of the cones K_i ; by $\mathfrak{N}_i(\mu_0, \rho_0)$ we shall denote the intersection $\mathfrak{N}(\mu_0, \rho_0) \cap K_i$. The problem of small solutions of equation (2) will be studied if it is possible to describe sufficiently completely the small solutions lying in certain cones K_i .

Let l be a certain eigenray of the operator PC , and let h be the unit vector of this ray. Introduce into consideration the number

$$\beta(h) = \left. \frac{d}{d\varepsilon} (PC(h + \varepsilon g), -\varepsilon h + g) \right|_{\varepsilon=0}, \quad (6)$$

where g is a vector in E_0 orthogonal to h ; $\beta(h)$ is easily computed. Since the angular coefficient of the eigenray l , by assumption, is a simple root of equation (4), $\beta(h) \neq 0$. If $\beta(h) < 0$, then the eigendirection l will be called attracting, and if $\beta(h) > 0$, repelling.

Denote by $K(l; \nu_1, \nu_2)$ ($\nu_1, \nu_2 > 0$) the set of such $x \in E$ that $(Px, h) > 0$, $|(Px, g)| \leq \nu_1(Px, h)$, $\|Qx\|_* \leq \nu_2(Px, h)$. It is easy to see that $K(l; \nu_1, \nu_2)$ is a cone that admits plastering. Denote by $T(l; \nu_1, \nu_2, \rho)$ the intersection of the cone $K(l; \nu_1, \nu_2)$ with the ball $\|x\|_* \leq \rho$.

Theorem 3 ⁽⁷⁾. *Let the ray l be attracting. Then there exist positive numbers t_1, t_2, t_3, t_4 such that, for every $x \in T(l; \nu_1, \nu_2; \rho)$, where $\nu_1 \leq t_1$, $\nu_2 \leq t_2\nu_1$, $\rho \leq t_3\nu_2$, the element Ax belongs to $K(l; \nu_1, \nu_2)$ together with the ball consisting of elements of the form $Ax + y$, where $\|y\|_* \leq t_4\nu_1\|x\|_*^2$.*

4. Let us introduce into consideration the operator

$$A_1x = QAx - PAx + \frac{2(PA_1x, Px)}{(Px, Px)}Px. \quad (7)$$

This operator is defined for $x \in E_0$.

Theorem 4. *Let the ray l be repelling. Then there exist positive numbers t_1, t_2, t_3, t_4 such that, for every $x \in T(l; \nu_1, \nu_2; \rho)$, where $\nu_1 \leq t_1$, $\nu_2 \leq t_2$, $\rho \leq t_3 \nu_2$, the element $A_1 x$ belongs to $K(l; \nu_1, \nu_2)$ together with the ball consisting of elements of the form $A_1 x + y$, where $\|y\|_* \leq t_4 \nu_1 \|x\|_*^2$.*

It is easy to verify that the solutions of equations (2) that do not lie in E^0 coincide with the solutions of the equations $A_1 x = (1 + \mu)x$.

5. From Theorems 3 and 4 it follows that

Theorem 5. *Let the operator A be completely continuous. Then each set $\mathfrak{R}_i(\mu_0, \rho_0)$ is nonempty and forms a continuous branch in a neighborhood of zero. Each continuous branch $\mathfrak{R}_i(\mu_0, \rho_0)$ is tangent at the zero point to the corresponding eigenray l_i .*

In particular, from a simple analysis of equation (4) and from Theorem 5 it follows that:

Theorem 6. *Let the rotation ⁽⁸⁾ of the two-dimensional vector field (3) on the unit circle be equal to -2 . Then $\mathfrak{R}(\mu_0, \rho_0)$ consists of 6 continuous branches, each of which is tangent at zero to one of the 6 eigenrays of the quadratic operator PC .*

6. From the preceding arguments it follows that $\mathfrak{R}(\mu_0, \rho_0)$ splits into either 2 or 6 branches $\mathfrak{R}_i(\mu_0, \rho_0)$. It can be shown that half of these branches consist entirely of solutions of equation (2) for positive μ , and the other half of solutions of equation (2) for negative μ .

7. The assertions given above can be applied, by the usual schemes, to the study of integral equations, problems on nonlinear oscillations, and others.

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