THE "QUASI-ISOTROPIC" APPROXIMATION OF GEOMETRICAL OPTICS

PHYSICS

1968

SovietRxiv

Abstract

Full Text

UDC 535.31

PHYSICS

Yu. A. KRAVTSOV

THE “QUASI-ISOTROPIC” APPROXIMATION OF GEOMETRICAL OPTICS

(Presented by Academician M. A. Leontovich on March 7, 1968)

1. As is known, in weakly anisotropic media, for which the off-diagonal components of the dielectric-permittivity tensor ε_{ik} and the differences between the diagonal components are small in comparison with $\mu \sim 1/kl \ll 1$, where l is the characteristic scale of variation of the properties of the medium, the geometrical-optics approximation in the form of independent normal waves is not applicable ([1]). In this case, however, one can construct a geometrical-optics solution of Maxwell' s equations based on the representation of waves in an isotropic medium, regarding the tensor $v_{ik} = \varepsilon_{ik} - \varepsilon\delta_{ik}$ as a perturbation $|v_{ik}| \ll \varepsilon$ (for ε one may take, for example, ${}^1/_3\,\mathrm{Sp}\,\varepsilon_{ik}$), and applying the theory developed by S. M. Rytov ([2,3]). The corresponding modification of the geometrical-optics method, for which S. M. Rytov proposed the name quasi-isotropic approximation, is set out below.

2. In order to be able to use the formal expansion of the field amplitudes in powers of $1/k$ (the wave number $k = \omega/c$ will be a large parameter of the problem), we assign the quantities v_{ik} the order of smallness $1/k$, putting $v_{ik} = \xi_{ik}/k$. Maxwell' s equations, which take the form

$$\mathrm{rot}\,\mathbf{H} + ik\varepsilon\mathbf{E} = -i\hat{\xi}\mathbf{E}, \qquad \mathrm{rot}\,\mathbf{E} - ik\mathbf{H} = 0, \tag{1}$$

differ from the equations for the field in an isotropic medium by the presence of the term $-i\hat{\xi}\mathbf{E}$, small in comparison with $k\varepsilon\mathbf{E}$.

Seeking an asymptotic solution of equations (1) in the same form as in an isotropic inhomogeneous medium, i.e., setting

$$\mathbf{E} = \sum_{m=0}^{\infty}(ik)^{-m}\mathbf{E}^{(m)}e^{ik\varphi}, \qquad \mathbf{H} = \sum_{m=0}^{\infty}(ik)^{-m}\mathbf{H}^{(m)}e^{ik\varphi},$$

one can verify that the phase φ satisfies, as in an isotropic medium, the eikonal equation $(\nabla\varphi)^2 = \varepsilon$, while the vector amplitudes $\mathbf{E}^{(0)}$ and $\mathbf{H}^{(0)}$ (the zeroth approximation of the method) are transverse to each other and to the direction of the wave normal $\mathbf{t} = \nabla\varphi/|\nabla\varphi|$. We can therefore put $\mathbf{E}^{(0)} = \Phi_1\mathbf{n} + \Phi_2\mathbf{b}$, $\mathbf{H}^{(0)} = \sqrt{\varepsilon}(\Phi_1\mathbf{b} - \Phi_2\mathbf{n})$, where $\mathbf{n}$ and $\mathbf{b}$ are unit vectors of the principal normal and binormal to the ray.

The compatibility conditions of the first-approximation equations, which have the form

$$\mathbf{H}^{(0)} \operatorname{rot} \mathbf{E}^{(0)} - \mathbf{E}^{(0)} \operatorname{rot} \mathbf{H}^{(0)} - i\mathbf{E}^{(0)}\hat{\xi}\mathbf{E}^{(0)} = 0, \tag{2}$$

$$\mathbf{H}^{(0)} \operatorname{rot} \mathbf{H}^{(0)} + \varepsilon\mathbf{E}^{(0)} \operatorname{rot} \mathbf{E}^{(0)} + i\mathbf{H}^{(0)}\hat{\xi}\mathbf{E}^{(0)} = 0,$$

are essentially two equations for determining Φ_1 and Φ_2. The solution

$$\mathbf{E} = \mathbf{E}^{(0)}e^{ik\varphi} = (\Phi_1\mathbf{n} + \Phi_2\mathbf{b})e^{ik\varphi}, \qquad \mathbf{H} = \mathbf{H}^{(0)}e^{ik\varphi} = \sqrt{\varepsilon}(\Phi_1\mathbf{b} - \Phi_2\mathbf{n}), \tag{3}$$

in which Φ_1 and Φ_2 are found from (2), is the principal term of the asymptotic series in the quasi-isotropic approximation.

3. Equations (2) can be written in another form if we introduce the notation $\chi = \ln(\sqrt{\varepsilon}\Phi^2)$, $\Phi^2 = \Phi_1^2 + \Phi_2^2$, $\theta = \operatorname{arctg}(\Phi_2/\Phi_1)$:

$$\frac{d\chi}{d\sigma} + \operatorname{div}\mathbf{t} = \frac{i}{2\sqrt{\varepsilon}}\left[(\xi_{nn} + \xi_{bb}) + (\xi_{nn} - \xi_{bb})\cos 2\theta + (\xi_{nb} + \xi_{bn})\sin 2\theta\right], \tag{4}$$

$$\frac{d\theta}{d\sigma} = \frac{1}{T} + \frac{i}{4\sqrt{\varepsilon}}\left[(\xi_{bn} - \xi_{nb}) + (\xi_{nb} + \xi_{bn})\cos 2\theta - (\xi_{nn} - \xi_{bb})\sin 2\theta\right]. \tag{5}$$

Here $\xi_{nn}, \xi_{nb}, \xi_{bn}, \xi_{bb}$ are the components of the tensor ξ_{ik} in the coordinate system whose axes coincide with the unit vectors of the natural trihedron $\mathbf{n}, \mathbf{b}, \mathbf{t}$; $d\sigma$ is an element of length, and T is the radius of torsion of the ray: $2/T = \mathbf{n} \operatorname{rot}\mathbf{n} + \mathbf{b} \operatorname{rot}\mathbf{b}$. It is essential that the angle $\theta = \operatorname{arctg}(\Phi_2/\Phi_1)$ can be found from equation (5) independently of the amplitude Φ. This angle is, in general, complex: $\theta = \theta' + i\theta''$. The real part θ determines the orientation of the polarization ellipse relative to the unit vectors $\mathbf{n}$ and $\mathbf{b}$, while the imaginary part determines the degree of elongation of the ellipse: the ratio of the minor axis to the major one is equal to $\operatorname{th}\theta''$.

In the limiting case of vanishingly small anisotropy ($v_{ik} = \xi_{ik}/k \to 0$), equations (4) and (5), of course, give the results obtained earlier for an isotropic medium; namely, from them one obtains the law of conservation of intensity $\operatorname{div}(\sqrt{\varepsilon}\Phi^2\mathbf{t}) = 0$ and the S. M. Rytov law for the rotation of the plane of polarization $d\theta/d\sigma = 1/T$ ([2,3]). It can be shown that, with increasing anisotropy (when $|v_{ik}| \gg \mu \simeq 1/kl$, but $|v_{ik}| \ll \varepsilon$), solution (3) passes into the sum of two so-called shortened normal waves, i.e., independent normal waves in whose amplitudes and phases terms of order $|v_{ik}|^2 \ll |v_{ik}|$ have been discarded. Owing to this, solution (3) is naturally joined to the normal waves (the latter were considered, for example, in ([1],[4],[5]) for the one-dimensional case and in ([6],[7]) for the three-dimensional case). As a result, the investigation of wave propagation at the beginning of an arbitrary three-dimensionally inhomogeneous anisotropic layer is considerably simplified. In this connection it should be clarified that the conclusion made in ([1],[4],[5]) concerning the inapplicability of geometrical optics in a weakly anisotropic medium (or at the beginning of an anisotropic layer) in fact pertains only to the approximation in the form of independent normal waves, but not to the quasi-isotropic approximation.

4. As an example of the use of equation (5), let us consider the problem of the change in polarization of a high-frequency field in a gyrotropic plasma, taking into account simultaneously both linear and quadratic magneto-optical effects. At high frequencies $v \equiv 4\pi e^2 N/m\omega^2 \ll 1$ and $|u| = eH_0/mc\omega \ll 1$ ($\mathbf{H}_0$ is a constant magnetic field). It is natural to take $1-v$ as ε. Calculating the components entering (5) of the tensor $\xi_{ik} = kv_{ik} = k(\varepsilon_{ik} - \varepsilon\delta_{ik})$ with the aid of expressions for ε_{ik}, available, for example, in ([1]), and retaining in them the terms linear and quadratic with respect to H_0, we obtain for θ the equation

$$\frac{d\theta}{d\sigma} = \frac{1}{T} + \frac{1}{2}kv\sqrt{u}\cos\alpha - \frac{i}{4}kvu\sin^2\alpha\sin 2(\theta+\psi), \tag{6}$$

where α is the angle between the vectors $\mathbf{t}$ and $\mathbf{H}_0$, and ψ is the angle between the principal normal to the ray $\mathbf{n}$ and the plane $(\mathbf{t}, \mathbf{H}_0)$.

Equation (6) describes the polarization of the field both in quasi-longitudinal ($\alpha \ll 1$) and in quasi-transverse ($|\pi/2 - \alpha| \ll 1$) propagation. In the case of quasi-longitudinal propagation the last term in (6) may be neglected, and we have

$$\theta(\sigma) = \theta(0) + \int_0^\sigma \frac{d\sigma}{T} + \theta_F(\sigma), \tag{7}$$

where

$$\theta_F(\sigma) = \frac{1}{2}k\int_0^\sigma v\sqrt{u}\cos\alpha\, d\sigma = \frac{2\pi e^3}{m^2c^2\omega^2}\int_0^\sigma NH_0\cos\alpha\, d\sigma.$$

Faraday rotation angle. Thus, for $\alpha \ll 1$ the resulting angle of rotation of the plane of polarization is determined by the torsion of the ray and by the Faraday effect.*

The last term in equation (6), quadratic with respect to H_0, describes the Cotton–Mouton effect. At not very large distances σ, explicit expressions for θ

can be obtained in the form of quadratures, taking into account that at high frequencies $u \ll \sqrt{u}$, and applying perturbation theory to (6). Taking (7) as the zero approximation, in the first approximation we have

$$\theta'(\sigma) = \operatorname{Re}\theta(\sigma) = \theta(0) + \int_0^\sigma \frac{d\sigma}{T} + \theta_F(\sigma),$$

$$\theta''(\sigma) = \operatorname{Im}\theta(\sigma) = -\frac{1}{4}\,k\int_0^\sigma uv\sin^2\alpha\,\sin\left[2\left(\theta'(\sigma)+\psi\right)\right]\,d\sigma. \quad (8)$$

The quantity $\operatorname{th}\theta'' \sim \theta''$, as stated above, determines the ratio of the minor semiaxis of the polarization ellipse to the major semiaxis, i.e., the degree of depolarization of the field. Formulas (8) may be useful for radio-astronomical and radiophysical applications.

The author expresses deep gratitude to S. M. Rytov, L. L. Goryshnik, and Yu. Ya. Yashin for discussion and valuable advice.

Radio Engineering Institute
Academy of Sciences of the USSR

Received
4 III 1968

CITED LITERATURE

[1] V. L. Ginzburg, *Propagation of Electromagnetic Waves in Plasma*, "Nauka," 1967.
[2] S. M. Rytov, DAN, **18**, 263 (1938).
[3] S. M. Rytov, *Proceedings of the P. N. Lebedev Physical Institute, Academy of Sciences of the USSR*, **2**, 1 (1940).
[4] J. A. Ratcliffe, *Magneto-Ionic Theory and Its Applications to the Ionosphere*, IL, 1962.
[5] K. G. Budden, *Radio Waves in the Ionosphere*, Cambridge, 1961.
[6] R. M. Lewis, *Arch. Rat. Mech. Anal.*, **20**, 191 (1965).
[7] Yu. A. Zaitsev, Yu. A. Kravtsov, Yu. Ya. Yashin, *Izv. Vyssh. Uchebn. Zaved., Radiofizika*, **11**, No. 12 (1968).

---

* This result was obtained in (7) by another, not quite legitimate route—from consideration of the superposition of two independent circularly polarized waves.

Note: Figure translations are in progress. See original paper for figures.