STABILIZATION OF THE INSTABILITY OF “TRAPPED” PARTICLES IN A DENSE PLASMA

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Abstract

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Corresponding Member of the USSR Academy of Sciences R. Z. SAGDEEV, A. A. GALEEV

STABILIZATION OF THE INSTABILITY OF “TRAPPED” PARTICLES IN A DENSE PLASMA

Taking “trapped” particles into account in the kinetic theory of plasma stability in toroidal magnetic traps has led to the discovery of a branch of low-frequency instabilities developing over many periods of oscillation of trapped particles between local magnetic mirrors ($^{1-3}$). The shear of the magnetic-field lines (“shear”), as it turned out, does not prevent the development of the instability. We shall show that in a sufficiently dense plasma, when the electron-ion collision frequency becomes sufficiently large, the growth rate of the drift-dissipative instability of trapped particles considered by B. B. Kadomtsev and O. P. Pogutse (^{2}) decreases, and then Landau damping on passing ions, omitted in (^{2}), can stabilize the instability.

As in ($^{1-3}$), we use a model of an axially symmetric magnetic field

$$\mathbf{H} = H_0\{(1-\varepsilon)\cos\vartheta\mathbf{e}_\zeta + \Theta\mathbf{e}_\vartheta\}, \qquad \varepsilon = r/R \ll 1, \quad \Theta \equiv i(r)r/2\pi R < \sqrt{\varepsilon}, \tag{1}$$

where r, ϑ are polar coordinates in the cross section of the toroidal tube; R is the major radius of the torus; ζ is the angular coordinate along the torus; $i(r)/2\pi$ is the magnitude of the rotational transform of the magnetic-field lines.

The equilibrium of a low-pressure plasma in such a system was studied in (^{4,5}). The distribution of passing particles $f_{uj}^{(0)}$ may, with sufficient accuracy, be taken as Maxwellian, while the distribution of trapped particles $f_{tj}^{(0)}$ is described by the expression

$$f_{tj}^{(0)}(\mu, v_\|, r, \vartheta) = \pi^{-3/2} v_{Tj}^{-3}\left[1 + \frac{\vartheta_\|}{\tilde{\omega}_{cj}\Theta}\left(\frac{d}{dr} + \frac{e_j}{T_j}\frac{d\Phi_0}{dr} - \frac{\mu H_0}{T_{jR}}\cos\vartheta\right)\right] \times$$

$$\times n_{0j}(r)\exp\left\{-\frac{e_j\Phi_0(r) + \mu H_\zeta}{T_j} - \frac{m_{jv_\|}^2}{2T_j}\right\}, \tag{2}$$

where $v_{Tj} = \sqrt{2T_j/m_j}$ is the thermal speed of particles of species j; $\tilde{\omega}_{cj} = e_{jH_\zeta}/m_{jc}$ is the cyclotron frequency of the particles in the toroidal magnetic field; $\mu = m^2_{jv_\perp}/2H_\zeta$ is the magnetic moment of a particle; $\Phi_0(r)$ is the potential of the unperturbed electric field.

The perturbations imposed on the equilibrium state of the plasma will be chosen in the form of electrostatic oscillations with potential

$$\Phi^{(1)}(\mathbf{r},t) = \Phi^{(1)}(\vartheta)e^{-i\omega t + il\zeta}, \qquad l = 0, \pm 1, \pm 2, \dots, \tag{3}$$

and special attention will be paid to perturbations with a frequency much smaller than the frequency of circulation of trapped particles along a closed trajectory,

$$\omega \ll \tau_j^{-1} \simeq v\Theta\sqrt{2\varepsilon}/r. \tag{4}$$

To describe such oscillations it is convenient to use the drift kinetic equation with the collision term in the Landau form. After li-

linearization the latter takes the form

$$\left\{-i\omega + \frac{dr}{dt}\frac{\partial}{\partial r} + \frac{d\vartheta}{dt}\frac{\partial}{\partial\vartheta} + \frac{ilv_\parallel}{R} - \frac{\mu H_0}{R}\sin\vartheta\frac{\partial}{\partial v_\parallel} - \widehat{St}\right\} f_j^{(1)} =$$

$$= -\frac{e_j}{m}\left\{\frac{\partial f_j^{(0)}}{\partial v_\parallel}\left(\frac{il}{R} + \theta\frac{\partial}{r\partial\vartheta}\right)\Phi^{(1)}(\vartheta) + \frac{1}{\omega_{cj}}\frac{\partial f_i^{(0)}}{\partial r}\frac{\partial\Phi^{(1)}(\vartheta)}{r\partial\vartheta}\right\}, \tag{5}$$

where the drift velocities of particles in the magnetic field (1) are determined by the expressions:

$$r\frac{d\vartheta}{dt} = \theta v_\parallel + v_E - \frac{\mu H_0\varepsilon}{m_j\omega_{cj}r}\cos\vartheta; \tag{6}$$

$$\frac{dr}{dt} = -\frac{\mu H_0\varepsilon}{m_j\omega_{cj}r}\sin\vartheta, \qquad \omega_{cj} = \frac{e_{jH}0}{m_{jc}}; \tag{7}$$

$$v_\parallel(r,\vartheta) = \left(2m_j^{-1}\left[E - \mu H_0(1-\varepsilon\cos\vartheta) - e_j\Phi_0(r)\right]\right)^{1/2}; \tag{8}$$

E is the particle energy; $v_E = -\frac{c}{H_0}\frac{d\Phi_0}{dr}$ is the velocity of the electric drift.

Because of the smallness of the number of trapped particles, the collision integral can be linearized and only derivatives with respect to the longitudinal velocity retained in it (6):

$$\widehat{St} = \sum_{j'} \frac{2\pi\lambda e_j^2 e_{j'}^2}{m_{jv}} \left(\eta_{j'} + \eta'_{j'} - \frac{\eta_{j'}}{2x_{j'}}\right) \frac{d^2}{\partial v_\parallel^2}, \tag{9}$$

where

$$\eta_j \equiv \eta(x_j) = 2\pi^{-1/2} \int_0^{x_j} e^{-t}\sqrt{t}\, dt, \qquad \eta' = \frac{d\eta(x)}{dx}, \qquad x_j = \frac{v^2}{v_{Tj}^2}.$$

In the first approximation in equation (5) one may neglect the slow change in the particle distribution and their drift. As a result we obtain

$$\Delta f_{tj} \equiv \left[f_{tj}^{(1)}(\vartheta) + e_j \Phi^{(1)}(\vartheta) f_{tj}^{(0)}/T_j\right] e^{-ilq\vartheta} = \text{const}, \tag{10}$$

where $q = -2\pi/\iota$ is the so-called safety factor in Tokamak-type systems $^{(2)}$.

In order to take into account the slow oscillations of the plasma, we must now average equation (5) over the period of the rapid oscillations of the trapped particles. It is first convenient to pass to the new variables μ, ϑ, and

$$\chi^2 = m_j \left(v_\parallel(r,0)\theta - v_E\right)^2 / 4\mu H_0 \varepsilon \theta^2, \tag{11}$$

in which the equation describing the rapid motion of trapped particles (6) takes the simple form

$$r\, d\vartheta/dt = \sigma v \theta \sqrt{2\varepsilon\,(2\chi^2 - 1 + \cos\vartheta)}, \qquad \sigma = \pm 1. \tag{12}$$

Averaging over the period of the rapid oscillations is equivalent to averaging over the angle ϑ according to the rule

$$\langle F\{\vartheta(t)\}\rangle = \frac{1}{4K(\chi)} \int_{-\vartheta_0}^{+\vartheta_0} \frac{F(\vartheta)\, d\vartheta}{\sqrt{\chi^2 - \sin^2\vartheta/2}}, \tag{13}$$

where $K(\chi)$ is the complete elliptic integral of the first kind. After averaging equation (5), we obtain the linear differential equation

$$\left\{\omega - \frac{l\langle v_{cj}\rangle}{R} - 2\frac{\nu_j}{\varepsilon} A(x_j) \frac{i}{K(\chi)} \frac{\partial}{\partial \chi^2} \left[E(\chi) - (1-\chi^2)K(\chi)\right] \frac{\partial}{\partial \chi^2}\right\} \Delta f_{tj} =$$

$$= \frac{e_j f_{tj}^{(0)}}{T_j}\left\{\omega - \frac{lq}{r}v_E - \frac{lq}{r}v_*\right\}\frac{1}{4K(\chi)}\int_{-\vartheta_0}^{\vartheta_0}\frac{\Phi(\vartheta')e^{-ilq\vartheta'}\,d\vartheta'}{\sqrt{\chi^2-\sin^2\vartheta'/2}}, \tag{14}$$

where the drift velocity of the particles along the ζ axis is found from the condition that the mean velocity $\langle d\vartheta/dt\rangle$ vanishes:

$$\langle v_\zeta^j\rangle = \frac{v_E}{\theta} - \frac{2\mu H_0 e}{m_j\omega_{cj}\theta r}\left\{\frac{E(\chi)}{K(\chi)} - \frac{1}{2} - 2\frac{d\ln i}{dr}\left(\frac{E}{K}-1+\chi^2\right)\right\},$$

$$v_*^j = \frac{v_{Tj}^2}{2\omega_{cj}}\frac{d\ln n(r)}{dr}; \tag{15}$$

$E(\chi)$ is the complete elliptic integral of the second kind. The collision frequencies of particles of species j are denoted by

$$\nu_j = \frac{16\sqrt{\pi}\,ne^4}{3\sqrt{m_j}\,v_{Tj}^3}, \qquad A(x_j) = \frac{3\pi^{1/2}}{4}\sum_{j'}\left\{\eta_{j'}+\eta'_{j'}-\frac{\eta_{j'}}{2x_{j'}}\right\}x_{j'}^{-3/2}. \tag{16}$$

We shall be interested in the case of frequent electron and rare ion collisions

$$\nu_e/\varepsilon \gg \omega \gg \nu_i/\varepsilon. \tag{17}$$

In this case, ion collisions may be completely neglected and equation (5) reduced to the algebraic form

$$f_{ti}^{(1)} = -\frac{e_i f_{ti}^{(0)}}{T_i}\left\{\Phi^{(1)}(\vartheta) - \frac{\omega - \frac{lq}{r}(v_*^j + v_E)}{\omega - \varepsilon\langle v_\zeta^i\rangle}\frac{e^{ilq\vartheta}}{4K(\chi)}\int_{-\vartheta_0}^{\vartheta_0}\frac{\Phi^{(1)}(\vartheta')e^{-ilq\vartheta'}}{\sqrt{\chi^2-\sin^2\vartheta'/2}}\,d\vartheta'\right\}. \tag{18}$$

In equation (5) for electrons, collisions play the dominant role. Taking into account that the distribution of trapped electrons in the field of a slow wave (with phase velocity much smaller than the electron thermal velocity) is Boltzmann, we solve equation (5) under the boundary conditions

$$f_{te}^{(1)}(\chi^2=1) = \frac{e\Phi^{(1)}}{T_e}f_{te}^{(0)}(\chi^2=1), \qquad \left.\frac{\partial f_{te}^{(1)}}{\partial\chi^2}\right|_{\chi^2=1} = 0. \tag{19}$$

As a result we find

$$f^{(1)}_{te} = \frac{e f^{(0)}_{te}}{T_e}\left\{\Phi^{(1)}(\vartheta) - \frac{\left[\omega - \frac{lq}{r}(v_E + v^j_*)\right]}{i\nu_j A(x_j)}\varepsilon e^{ilq\vartheta}\times\right.$$

$$\left.\times\int_{\chi^2}^{1}\frac{d\alpha^2}{2\left[E(\alpha)-(1-\alpha^2)K(\alpha)\right]}\int_0^{\alpha^2} d\beta^2\int_{-\vartheta_0}^{+\vartheta_0}\frac{\Phi(\vartheta')e^{-ilq\vartheta'}}{\sqrt{\beta^2-\sin^2\vartheta'/2}}\frac{d\vartheta'}{4}\right\}. \quad (20)$$

Finally, the expression for the phase-volume element of trapped particles is obtained by using the expression for the longitudinal velocity $v_\parallel(\mu,\chi^2,\vartheta)$ in the variables μ,χ^2,ϑ, following from equations (6) and (14):

$$2\pi\frac{H_\zeta\,d\mu}{m_j}\,dv_\parallel(\mu,\chi^2,\vartheta) = 2\pi v\left[\left(v^2+\frac{v_E^2}{\theta^2}\right)\varepsilon\right]dv\,\frac{d\chi^2}{\sqrt{2[\chi^2-\sin^2\vartheta/2]}}\sum_\sigma. \quad (21)$$

We shall give the plasma quasineutrality equation, which serves to determine the spectrum of frequencies and amplitudes of the arising oscillations, only for the case of a weak equilibrium electric field, when terms of order $[v_E/\theta v_{Ti}] \ll 1$ may be neglected. In addition, expanding the function $\Phi^{(1)}(\vartheta)$ in a Fourier series,

$$\Phi^{(1)}(\vartheta) = \sum_{m=-\infty}^{+\infty}\widetilde{\Phi}_m e^{im\vartheta} \quad (22)$$

and substituting expressions (18) and (20) into the quasineutrality condition, we reduce the latter to a system of algebraic equations for the coefficien-

of $\widetilde{\Phi}_m$

$$\sum_{s'}\left\{\delta_{ss'} - \frac{\omega^e_{*t}}{\omega'}H_{ss'} - i\omega^e_{*t}\langle\nu_e^{-1}\rangle\nu_{ss'} - i2\pi^{1/2}\frac{\omega^e_{*t}\omega {a'}^2\tau_i^3}{|s'|^3}A_{ss'}\right\}\Phi_{s'} = 0, \quad (23)$$

where

$$\omega' = \omega - \frac{lq}{r}v_E, \qquad \omega^j_{*t} = \frac{2\sqrt{2\varepsilon}}{\pi}\frac{lq}{r}v^j_*\bigg/\left(1+\frac{T_e}{T_i}\right), \qquad \tau_i = \frac{r}{v_{Ti}\theta\sqrt{2\varepsilon}},$$

$$\langle\nu_e^{-1}\rangle \equiv \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x_e}\frac{x_e^{1/2}\,dx_e}{\nu_e A(x_e)} \simeq 1.2\,\nu_e^{-1}, \qquad \Phi_{ml-q} \equiv \widetilde{\Phi}_m,$$

$$H_{ss'} = \frac{\pi^2}{8} \int_0^1 \frac{dt}{K(\sqrt{t})} P_{s-1/2}(1-2t) P_{s'-1/2}(1-2t), \qquad s \equiv m - lq,$$

$$\nu_{ss'} = \frac{\pi^2}{16} \int_0^1 d\chi^2 \left[\int_0^{\chi^2} P_{s-1/2}(1-2t)\, dt \right] \frac{1}{E(\chi) - (1-\chi^2) K(\chi)} \left[\int_0^{\chi^2} P_{s'-1/2}(1-2t)\, dt \right],$$

$$A_{ss'} = \frac{\pi^2}{8} \int_1^\infty \frac{d\chi^2}{\chi^4} K^2 \left(\frac{1}{\chi} \right) \Pi_s(\chi) \Pi_{s'}(\chi) f^{(0)} \left(\frac{\omega' \tau_i K(1/\chi)}{|s'|\chi} \right),$$

$$P_{s-1/2}(1-2\chi^2) = \frac{2}{\pi} \int_0^a \frac{\cos 2s\alpha}{\sqrt{\chi^2 - \sin^2 \alpha}}\, d\alpha, \qquad \Pi_s(\chi) = \frac{2}{\pi} \int_0^{\pi/2} \frac{\cos 2s\alpha}{\sqrt{1 - \chi^{-2} \sin^2 \alpha}}\, d\alpha.$$

Here the last term takes into account the resonant interaction of passing ions with slow oscillations; moreover, the phase velocity of the long-wavelength oscillations turns out to be so small that only a small number of slow ions with velocities $v/v_{Ti} \sim \omega\tau_i/|s| < 1$ are in resonance with the wave. Restricting ourselves in solving equation (23) to a system of third-order equations (for $\Phi_{0,\pm 1}$) and using the numerical values of the matrix elements $H_{ss'}$, $\nu_{ss'}$, $A_{ss'}$ for $m \equiv m^{(0)} \simeq lq$ and $m = m^{(0)} \pm 1$: $H_{00} = 1$, $H_{01} = 1/3$, $H_{11} \simeq 0.28$; $\nu_{00} = 4/9$, $\nu_{01} \simeq 0.142$, $\nu_{11} \simeq 0.132$, $A_{00} \simeq 9.1$, $A_{01} \simeq 1.9$, $A_{11} \simeq 0.68$, we find the eigenvalue of the oscillation frequency:

$$\omega = 1.3\, \omega^e_{*t} \left\{ 1 + i \left[0.96\, \frac{\omega^e_{*t}}{\nu_e} \varepsilon - 18\, \frac{\omega^{e3}_{*t} \tau_i^3}{|s^{(0)}|^3} \right] \right\}. \tag{24}$$

We see that Landau damping has the weakest stabilizing effect on the long-wavelength oscillations, which, however, are suppressed by weak collisions of the trapped ions under the condition

$$\nu_i \nu_e / \varepsilon^2 > [\omega^e_{*t}]^2. \tag{25}$$

Combining (24) and (25), we find the stability criterion

$$\frac{\nu_i}{\varepsilon} \tau_i > 0.37 \left(\frac{m_e}{m_i} \right)^{1/3} \frac{T_e}{T_i} |m^{(0)} - lq|. \tag{26}$$

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CITED LITERATURE

1. B. B. Kadomtsev, *Letters to the Editor of JETP*, **4**, 15 (1966).

2. B. B. Kadomtsev, O. P. Pogutse, JETP, **51**, 1734 (1966).

3. A. A. Galeev, R. Z. Sagdeev, H. V. Wong, *Phys. Fluids*, **10**, 1535 (1967).

4. H. L. Berk, A. A. Galeev, *Phys. Fluids*, **10**, 441 (1967).

5. A. A. Galeev, R. Z. Sagdeev, JETP, **53**, 348 (1967).

6. B. A. Trubnikov, in: *Problems of Plasma Theory*, **1**, 1963, p. 98.

Note: Figure translations are in progress. See original paper for figures.