ON A BOUNDARY-VALUE PROBLEM FOR AN EQUATION OF MIXED PARABOLIC-HYPERBOLIC TYPE

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Abstract

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ON A BOUNDARY-VALUE PROBLEM FOR AN EQUATION OF MIXED PARABOLIC-HYPERBOLIC TYPE

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Consider the equation

$$0 = \begin{cases} k(y)u_{xx} + u_{yy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u, & y < 0, \\ u_{xx} + \alpha(x,y)u_x + \beta(x,y)u_y + \gamma(x,y)u, & y > 0, \end{cases} \qquad (1)$$

in a simply connected mixed domain D of the plane of the independent variables x, y, bounded by the segments AA_0, A_0B_0, BB_0 of the straight lines $x = 0$, $x = 1$, $y = y_0 > 0$, respectively, and by the real characteristics $AC : \sqrt{-k}\, dy + dx = 0$, $BC : \sqrt{-k}\, dy - dx = 0$ of equation (1), issuing from the points $A(0, 0)$, $B(1, 0)$. Let $D_1(D_2)$ be the parabolic (hyperbolic) part of the mixed domain D.

The following assumptions are made concerning the coefficients of equation (1): $k(y)$ is a continuously differentiable and monotonically increasing function in D_2, with $k(0) = 0$; a, b, and c belong to the space $C^1(\overline{D}_2)$ and are connected by the relations (see ([1]))

$$\delta(\sqrt{-k}) + a + b\sqrt{-k} < 0,$$

$$\delta\left(\frac{\delta\sqrt{-k}}{\sqrt{-k}} + \frac{a + b\sqrt{-k}}{\sqrt{-k}}\right) + \frac{1}{2k}\left[\delta(\sqrt{-k}) + a + b\sqrt{-k}\right]$$

$$\times \left[\delta(\sqrt{-k}) + a - b\sqrt{-k}\right] - 2c \le 0, \qquad c \le 0,$$

where $\delta \equiv \partial\backslash\partial y + \sqrt{-k}\, \partial\backslash\partial x$; $\alpha, \beta, \gamma \in C(\overline{D}_1)$ and, moreover, $\beta < 0$, $\gamma \le 0$.

By a solution of equation (1) that is *regular in the domain* D we shall mean a function

$$u(x, y) \in C(\overline{D}) \cap C^1(D) \cap C^2(D_1 \cup D_2)$$

and satisfying equation (1) in $D_1 \cup D_2$.

For equation (1), the direct analogue of the well-known Tricomi problem ([2]) is the following

Problem T. Find a solution $u(x, y)$, regular in D, of equation (1), satisfying the boundary conditions:

$$u(0, y) = \varphi_0(y), \qquad u(1, y) = \varphi_1(y), \qquad 0 \le y \le y_0, \tag{2}$$

$$u\big|_{AC} = \psi(x), \tag{3}$$

where $\varphi_0(y), \varphi_1(y) \in C\ (0 \le y \le y_0)$, $\psi(x) \in C^4\ (0 \le x \le 1/2)$, $\varphi_0(0) = \psi(0) = \varphi_1(1) = 0$.

Lemma. Suppose: 1) $u(x, y)$ is a regular solution of problem T when $\psi(x) \equiv 0$; 2) the derivative of the function $u(x, y)$ in the direction of the characteristics of the family $\sqrt{-k}\,dy + dx = 0$ exists and is continuous in $\overline{D}_2 \setminus A \setminus B$. Then the positive maximum (negative minimum) of the function $u(x, y)$ in $\overline{D}_2$ is attained at some point $(\xi, 0) \in AB$, and at this point $u_y > 0$ ($u_y < 0$).

This lemma is proved in the same way as in ([1,3]), where the case $k(y) = y$, $a(x, y) \equiv 0$ was considered, while $b(x, y)$ and $C(x, y)$ are connected in a special way.

Let $u(x, y) \in C^1(D_1 \cup A_0B_0)$ and satisfy the conditions of the lemma. Then the positive maximum and negative minimum of $u(x, y)$ in $\overline{D}_1$ are attained on $\overline{AA_0} \cup \overline{BB_0}$.

The validity of this extremum principle follows from the lemma just given and the known extremum principle for parabolic equations ([4]).

From the extremum principle follows the uniqueness of the solution $u(x, y)$ of problem T, if $u(x, y) \in C^1(D_1 \cup A_0B_0)$.

We shall prove the existence of a solution of problem T, restricting ourselves to the case $a = b = c = \alpha = \gamma = 0,\ \beta = -1,\ k(y) = y$.

Let there exist a solution $u(x, y)$ of problem T, and let $u(x, 0) = \tau(x)$, $u_y(x, 0) = \nu(x)$. Then from equation (1) it follows directly that $\tau(x)$ and $\nu(x)$ on AB will be connected by the following relation, brought from the domain D_1:

$$\tau''(x) = \nu(x), \qquad 0 < x < 1,$$

or

$$\tau(x) = \int_0^x (x - t)\nu(t)\,dt + x\int_0^1 (t - 1)\nu(t)\,dt. \tag{4}$$

Solving the Darboux problem for equation (1) in the domain D_2 with data (3) and $\nu(x)$ on AB, we find ([5]) that these functions are also connected by the relation

$$\tau(x) = \gamma_0 \int_0^x \frac{\nu(t)\,dt}{(x-t)^{1/3}} + \Psi(x), \qquad 0 < x < 1, \tag{5}$$

where $4\pi^2\gamma_0 = 3^{2/3}\Gamma^3(1/3)$,

$$\Psi(x) = \frac{\Gamma(1/6)}{\Gamma(5/6)\Gamma(1/3)} x^{-1/6} \int_0^x \left[\psi'\left(\frac{\eta}{2}\right) + \frac{\psi(\eta/2)}{6\eta}\right] \eta^{1/3}(x-\eta)^{-1/6}\,d\eta. \tag{6}$$

Since

$$\psi(x) = x\int_0^1 \psi'(tx)\,dt, \qquad 0 \le x \le \frac{1}{2},$$

from (6) it is easy to see that

$$\Psi(x) \in C^3(0 \le x \le 1), \qquad \Psi(x) = O(1)x. \tag{7}$$

Eliminating $\tau(x)$ from the system (4)–(5), we obtain

$$\int_0^x \frac{\gamma_0 - (x-t)^{4/3}}{(x-t)^{1/3}}\nu(t)\,dt = x\int_0^1 (t-1)\nu(t)\,dt - \Psi(x). \tag{8}$$

Obviously, equation (8) is equivalent to problem T.

Applying to both sides of the integral equation (8) the operator (see ([6]))

$$A\mu(x) \equiv \frac{\partial}{\partial x}\int_0^x \frac{\mu(t)\,dt}{(x-t)^{2/3}}, \tag{9}$$

after some transformations we find

$$\frac{2\pi}{\sqrt{3}}\gamma_0\nu(x) - 3\int_0^x (x-t)^{1/3}\nu(t)\,dt = \int_0^1 (t-1)\nu(t)\,dt\,Ax - A\Psi(x),$$

or

$$\nu(x) - \lambda\int_0^x (x-t)^{1/3}\nu(t)\,dt = \lambda x^{1/3}\int_0^1 (t-1)\nu(t)\,dt + \Phi(x), \tag{10}$$

where

$$\lambda = \frac{3\sqrt{3}}{2\pi\gamma_0}, \qquad \Phi(x) = -\frac{\lambda}{3} A\Psi(x). \tag{11}$$

Let $\Gamma(x, t, \lambda)$ be the resolvent of the kernel $(x - t)^{1/3}$ of the Volterra operator in the left-hand side of equation (10). It is not difficult to verify that

$$\Gamma(x, t, \lambda) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}\Gamma^n(4/3)}{\Gamma(4n/3)} (x - t)^{(4n-3)/3}. \tag{12}$$

After the Volterra integral operator is inverted, which is legitimate by virtue of (7) and (11), equation (10) takes the form

$$v(x) - \lambda \int_0^1 k(x, t) v(t)\, dt = f(x), \tag{13}$$

where

$$k(x, t) = (t - 1) \left[x^{1/3} + \lambda \int_0^x s^{1/3} \Gamma(x, s, \lambda)\, ds \right], \tag{14}$$

$$f(x) := \Phi(x) + \lambda \int_0^x \Gamma(x, t, \lambda) \Phi(t)\, dt. \tag{15}$$

From (11), taking (7) into account, we have

$$\Phi(x) = -\frac{\lambda}{3} \int_0^x \frac{\Psi'(t)\, dt}{(x - t)^{2/3}},$$

therefore $\Phi(x) \in C(0 \le x \le 1) \cap C^2(0 < x < 1)$, and at the point $x = 0$ it vanishes to order not less than 1/3. Obviously, by virtue of (12) and (15), the right-hand side $f(x)$ of equation (13) possesses the same properties.

From (14) and (12) we conclude:

1) $k(x, t) \in C(0 \le x, t \le 1)$, $k(x, t) = x^{1/3} k^*(x, t)$, where $k^*(x, t) \in C(0 \le x, t \le 1)$;

2) $k(x, t)$ is twice continuously differentiable with respect to x in the square $0 < x, t < 1$.

Thus, if solutions $v(x)$ of the Fredholm integral equation of the second kind (13) exist, then they belong to the class
$C(0 \leq x \leq 1) \cap C^2(0 < x < 1)$ and at $x = 0$ vanish to order not less than 1/3. From relation (5) it follows that, for such $v(x)$, the function
$\tau(x) \in C(0 \leq x \leq 1) \cap C^2(0 < x < 1)$ and at $x = 0$ vanishes to order not less than one.

The solutions $u(x, y)$ of problem T in the domain D_2 (if they exist) are representable by Darboux' s formula

$$u(x,y) = \frac{\Gamma(1/3)}{\Gamma^2(1/6)} \int_0^1 \tau\Big[x + \frac{2}{3}(-y)^{3/2}(2t-1)\Big]\,[t(1-t)]^{-5/6}\,dt +$$

$$+\frac{\Gamma(5/3)}{\Gamma^2(5/6)}\,y \int_0^1 v\Big[x + \frac{2}{3}(-y)^{3/2}(2t-1)\Big]\,[t(1-t)]^{-1/6}\,dt,$$

whence it follows that $u(x, y) \in C^1(\overline{D}_2)$.

It is now not difficult to see the equivalence (in the sense of solvability) of problem T to the integral equation (13), and that the solution $u(x, y)$ of this problem belongs to the class of functions for which the extremum principle has been proved.

From the uniqueness of the solution of problem T there follows the unconditional solvability of the Fredholm integral equation of the second kind (13), and, consequently, of problem T.

Let now D_1 be a simply connected domain of the half-plane $y > 0$, whose boundary contains the segment AB of the axis $y = 0$, and let $D = D_1 \cup D_2 \cup AB$. In the general case, in proving the existence of a solution of problem T in the domain D, no fundamental difficulties arise if, in addition, the domain D_1 and the functions $a, b, c, \alpha, \beta, \gamma$ are such that in the domain D_1 the basic boundary-value problem for equation (1) (4) is solvable, and in D_2 the Darboux problem (5) is solvable.

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Note: Figure translations are in progress. See original paper for figures.