ON SOLVABLE VARIETIES OF GROUPS

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Abstract

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ON SOLVABLE VARIETIES OF GROUPS

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A variety of groups is called **solvable** if all its groups are solvable. Such is, in particular, every product $\mathfrak{N}_k\mathfrak{A}$ of the variety of all nilpotent groups of class $\leqslant k$ and the variety $\mathfrak{A}$ of all abelian groups (recall that the product $\mathfrak{U}\mathfrak{V}$ of varieties $\mathfrak{U}$ and $\mathfrak{V}$ consists of all possible extensions of groups from $\mathfrak{U}$ by means of groups from $\mathfrak{V}$). We shall call two-step solvable groups, and varieties consisting of metabelian groups, **metabelian**; all metabelian groups form the variety $\mathfrak{S}_2 = \mathfrak{A}^2$.

Our main result is the following.

Theorem 1. *Let $\mathfrak{B}$ be a variety contained in $\mathfrak{N}_k\mathfrak{A}$ for some $k \geqslant 1$, but not containing the variety $\mathfrak{S}_2$ of all metabelian groups. Then every group $G \in \mathfrak{B}$ has a series of normal subgroups*

$$G \supseteq U \supseteq T \supseteq 1,$$

where G/U and T have finite exponents, and U/T is nilpotent. In particular, in the variety $\mathfrak{B}$ every torsion-free group with a finite number of generators has a nilpotent normal subgroup of finite index.

Lemma 1. *Let G be a free group of rank 2 of some variety $\mathfrak{B} \subseteq \mathfrak{S}_2$, $\mathfrak{B} \neq \mathfrak{S}_2$. If the group G is torsion-free, then in it, for some $n > 0$, the relation*

$$[u, \underbrace{x, x, \dots, x}_{n}, \underbrace{x^2, \dots, x^2}_{n-1}, \dots, \underbrace{x^n}_{1}] = 1$$

holds for every $u \in G'$ and every x which is a free generator modulo G' (in other words, for every x from which, modulo G', no root of any degree > 1 can be extracted).

Proof. If x, y are free generators of the group G, then G' is a cyclic G/G'-module with generator $[x, y]$. Since G is not a free metabelian group (see (1)), G' is not a free module. Denote by O the annihilator of the module G' in the integral group ring $Z(G/G')$. An element lies in O if and only if it annihilates some generator of G/G', for example $[x, y]$. The ideal O has the following properties: 1) if $mf(x, y) \in O$, where $f(x, y) \in Z(G/G')$ and m is an integer, then $f(x, y) \in O$–this follows from the fact that the group G' is torsion-free; 2) if $x' = x^\alpha y^\beta,\ y' = x^\gamma y^\delta$,

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \pm 1$$

and $f(x, y) \in O$, then $f(x', y') \in O$. Indeed, in this case there exist free generators $\bar{x}, \bar{y}$ of the group G such that $\bar{x} \equiv x',\ \bar{y} \equiv y' \pmod{G'}$; therefore $f(x', y')$ annihilates $[\bar{x}, \bar{y}]$, which is a generator of the module G'.

We first show that O contains some polynomial $f(y)$. Obviously, O contains some polynomial $f(x, y)$. Arrange it according to powers of x:

$$f(x, y) = f_0(y) + f_1(y)x + \dots + f_n(y)x^n.$$

Let $f(x, y)$ be chosen so that n is minimal. Then, making the substitution $x' = xy,\ y' = y$, we obtain

$$f(x', y') = f_0(y) + f_1(y)yx + \dots + f_n(y)y^n x^n \in O.$$

Then $x^{-1}(f(x', y') - f(x, y)) \in O$ and has smaller degree with respect to x. Hence $n = 0$, and O contains some polynomial $f(y) \neq 0$. Let $f(y) = a_0 + a_1 y + \dots + a_n y^n$. Making n substitutions $x' = x,\ y' = yx^i,\ i = 1, \dots, n$, and substituting in $f(y)$, we obtain

$$\begin{gathered} a_0 + a_1 y + \dots + a_n y^n = f(y) \in O, \\ a_0 + a_1 yx + \dots + a_n y^n x^n = f(xy) \in O, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_0 + a_1 yx^n + \dots + a_n y^n x^{n^2} = f(x^{ny}) \in O. \end{gathered} \tag{1}$$

If this system of equalities is regarded as a linear system relating $a_0, a_1 y, \dots, a_n y^n$, then its determinant is the Vandermonde determinant $W(1, x, \dots, x^n)$. Hence we obtain that, for any $0 \leq j \leq n$,

$$a_j y^{jW}(1, x, \dots, x^n) \in O.$$

By the invertibility of y and property 1 of the ideal O, we have

$$W(1, x, \dots, x^n) \in O.$$

Multiplying $W(1, x, \dots, x^n)$ by a suitable negative power of x, we obtain

$$(x-1)^n(x^2-1)^{n-1}\cdots(x^n-1)\in O, \tag{2}$$

and from property 2 of the ideal O it follows that here, as x, one may take any element from which roots are not extracted in G/G'. The inclusion obtained, translated into the language of the group G, is precisely the assertion of the lemma.

Lemma 2. Let A be a certain module over a free abelian group B of finite rank, and suppose that A, under addition, is a free abelian group of finite rank. If

$$a(x-1)^n(x^2-1)^{n-1}\cdots(x^n-1)=0$$

for every $a\in A$ and every free generator x of the group B, then for some s,t

$$A(B^s-1)^t=0.$$

Proof. We proceed by induction on the rank of the group A. If A is cyclic, then $A(B^2-1)=0$, since every automorphism of the group A has order 2. Suppose that our assertion has been proved in the case where A has rank $<r$, and let A have rank r. Suppose that A has a submodule $A_1\neq 0$ of rank $<r$. We may assume that the group A/A_1 has no elements of finite order, for these can be adjoined to A_1. By the induction hypothesis, $(A/A_1)(B^{s_1}-1)^{t_1}=0,\ A_1(B^{s_2}-1)^{t_2}=0$. Then for the module A it suffices to take $s=s_1s_2,\ t=t_1+t_2$. Suppose now that A has no submodules of smaller rank. If x is some free generator of the group B, then every submodule $A(x-1)^{\alpha_1}(x^2-1)^{\alpha_2}\cdots(x^n-1)^{\alpha_n}$, $\alpha_i\leq n+1-i$, is either zero or contains some submodule of the form mA. By the condition of the lemma, for some $i>0$, $mA(x^i-1)=0$, whence $A(x^i-1)=0$.

Since the group B has finite rank, $A(x^i-1)=0$ will hold for every x from a fixed system of generators of the group B, for some $i>0$. But then from the equality $(xy)^i-1=(x^i-1)(y^i-1)+(x^i-1)+(y^i-1)$ it follows that $A(B^i-1)=0$. The lemma is proved.

Lemma 3. The group G of Lemma 1 has a nilpotent normal subgroup of finite index.

Proof. The group G', isomorphic as a module to $Z(G/G')/O$, has a finite number of generators.

For example, for the additive group $Z(G/G')/O$, generators are $x^iy^j,\ 0\leq i,j<r$, where r is the degree of the polynomial in (2). This follows from the fact that,

that the polynomial (2) and the analogous one written for y have leading and trailing coefficients ±1. Hence Lemma 2 is applicable modulo G'. Thus,

$$[G', \underbrace{G^s, G^s, \dots, G^s}_{t \text{ times}}] = 1 \tag{3}$$

whence G^s is nilpotent of class $\leq t+1$. Since the group G is solvable, G/G^s is finite. The lemma is proved.

Lemma 4. *Let the abelian torsion-free group A be a module over an abelian group B, and suppose that for every $x \in B$, $A(x^s-1)^t = 0$. Then*

$$A(B^s-1)^{2t} = 0.$$

The proof is analogous to one argument of K. Gruenberg (2) and is carried out by induction on t. For $t=1$ the assertion is obvious. Suppose it has already been proved for t; we prove it for $t+1$. Clearly, in the endomorphism ring of the group A, for $x, y \in B$ we have

$$0 = (x^s y^s - 1)^{t+1} = \big((x^s-1)(y^s-1) + (x^s-1) + (y^s-1)\big)^{t+1} =$$

$$= \sum_{i+j \geq t+1} a_{ij}(x^s-1)^i(y^s-1)^j.$$

Multiplying this equality by $(x^s-1)^{t-1}$ and taking into account the condition of the lemma for $t+1$, applied both to x and to y, we obtain

$$a_{1t}(x^s-1)^t(y^s-1)^t = 0,$$

or, since A is torsion-free,

$$(x^s-1)^t(y^s-1)^t = 0. \tag{4}$$

Now consider the aggregate A_1 of linear combinations of elements of the form $a(x^s-1)^t$, for all possible $a \in A,\ x \in B$. If in A/A_1 there are elements of finite order, adjoin them to A_1; then we obtain, obviously, a submodule A_2 of the module A, and in A/A_2 and, by virtue of (4), in A_2 the condition of the lemma for t is fulfilled. The rest is obvious.

Lemma 5. *Let the group G be the same as in Lemma 1, but of arbitrary rank. Then for some $s > 0$, G^s is nilpotent.*

Indeed, from Lemma 3 it follows that in every group of the variety $\mathfrak{V}$ the identity

$$[x^s, \underbrace{y^s, \dots, y^s}_{t}] = 1$$

holds for some $t > 0$, and then it suffices to use Lemma 4.

Lemma 6. *Suppose that in a certain group S there are a nilpotent normal torsion-free divisor N and the factor group $S/I(N')$ by the isolator of the commutator subgroup N' in N. Then the group S is also nilpotent.*

Apparently this lemma is known. Its proof is not difficult and we omit it.

Proof of Theorem 1. It is clear that it suffices to prove the theorem for a $\mathfrak{V}$-free group G of countable rank. One may also assume that the variety $\mathfrak{V}$ has exponent zero. Then all elements of finite order lie in G' and, by virtue of the nilpotency of G', form a subgroup P. Consider the group $H = G/P$. By the complete characteristicity of P in G, H is free in some variety $\mathfrak{B} \subset \mathfrak{N}_c\mathfrak{A}$. The group $H/I(H'')$ satisfies the condition of Lemma 5; therefore for some $s > 0$, $(H/I(H''))^s$ is nilpotent. Applying Lemma 6 to the case where $N = H'$, and S is the complete preimage of $(H/I(H''))^s$ in H, we obtain that H^s is nilpotent of class c. Take in G $U = G^s$, and as T the subgroup generated by all commutators $[g_1^s, g_2^s, \dots, g_{c+1}^s]$, $g_1, \dots, g_{c+1} \in G$. Since H^s is nilpotent of class c, all these commutators have finite orders, and these orders are divisors of the order of the element $[x_1^s, x_2^s, \dots, x_{c+1}^s]$,

where $x_1, x_2, \dots, x_{c+1}, \dots$ is a system of free generators of the group G. Obviously, in a nilpotent group (in our case in G') any set of elements of bounded orders generates a subgroup of positive exponent. Thus, T has positive exponent. In the group G/T the identity $[x_1^s, x_2^s, \dots, x_{c+1}^s] = 1$ holds; hence (by induction on c) it is not hard to obtain that $(G/T)^s = G^s/T$ is nilpotent of class $\leq c$. The theorem is proved.

Remark. Obviously, one may also take $G^s \cdot G'$ as U, so that one may assume that G/U is abelian of finite exponent.

Theorem 2. *Let G be a polycyclic group such that, in the variety* var(G) *generated by G, the free group on two generators is polycyclic. Then G has a nilpotent normal subgroup of finite index. Obviously, the converse is true for any number of generators.*

By a theorem of A. I. Mal' cev ([3]), for some s, G^s has a nilpotent commutant. We shall assume that the group G itself has a nilpotent commutant. Moreover, one may assume that G is torsion-free (otherwise it would have a torsion-free normal subgroup of finite index). If var(G) contains $\mathfrak{C}_2$, then the hypothesis of the theorem is violated. Hence Theorem 1 is applicable.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.