THE SECOND APPROXIMATION IN THE THEORY OF ASYMPTOTIC DAMPING OF DISTURBANCES IN A SUPERSONIC FLOW OF A VISCOUS HEAT-CONDUCTING GAS

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Abstract

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THE SECOND APPROXIMATION IN THE THEORY OF ASYMPTOTIC DAMPING OF DISTURBANCES IN A SUPERSONIC FLOW OF A VISCOUS HEAT-CONDUCTING GAS

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In paper ([1]) an asymptotic picture was constructed, in the first approximation, of the flow past bodies of revolution by a uniform supersonic stream of a viscous heat-conducting gas. It turned out that the equations describing the flow in the first approximation coincide with the equations for the propagation of acoustic pulses in nonstationary problems with cylindrical symmetry ([2]). In the present paper the equations of the second approximation are derived, valid both for transonic flows and for supersonic flows. On their basis the properties of supersonic flows are investigated. Gas motions with critical velocity were considered in papers ([3], [4]).

We take the equations of continuity, Navier–Stokes, and heat transfer, to which the gas motion will be subject, in the form in which they are given in the well-known monograph ([5]). Here x and r are a system of cylindrical coordinates, the x-axis of which coincides with the axis of symmetry of the body; v_x and v_r are the components of the velocity vector along these axes; ρ is the density; p is the pressure; s is the specific entropy; T is the temperature; λ_1 is the coefficient of viscosity; λ_2 is the coefficient of second viscosity; k is the coefficient of thermal conductivity. The entropy and temperature are eliminated from this system by means of the thermodynamic relations

$$ds = \frac{c_p}{\alpha\rho a^2 T}(dp - a^2 d\rho), \qquad dT = \frac{1}{\alpha\rho a^2}(\chi dp - a^2 d\rho) \tag{1}$$

$$\left(\alpha = \frac{1}{V}\left(\frac{dV}{dT}\right)_p, \qquad a^2 = \left(\frac{dp}{d\rho}\right)_s, \qquad V = \frac{1}{\rho}, \qquad \chi = \frac{c_p}{c_v}\right),$$

which relate the quantities p, ρ, T, and s to one another. As independent variables the density and pressure are taken. In the latter relations $V = 1/\rho$

denotes the specific volume, α the coefficient of thermal expansion, a the adiabatic speed of sound, c_p the specific heat at constant pressure, and c_v the specific heat at constant volume. In what follows we shall assume that the coefficients of viscosity and thermal conductivity depend only on the temperature.

Let us assume that the values of all gas parameters in the region of space under consideration deviate only slightly from the corresponding values in the incident stream. The undisturbed flow is uniformly steady; the speed of the particles in it exceeds the speed of sound in magnitude and is directed along the x-axis. In the limiting case it may coincide with the critical speed.

We introduce a new system of coordinates n and τ, connected with the characteristics of the undisturbed stream, which are obtained by replacing the Navier –Stokes and heat-transfer equations by the Euler equations and the equation of conservation of entropy in a particle:

$$x = n \sin A_\infty + \tau \cos A_\infty, \qquad r = -n \cos A_\infty + \tau \sin A_\infty$$

$$(\sin A_\infty = 1/M_\infty, \quad M_\infty = v_\infty / a_\infty).$$

The angle and Mach number of the incident flow are denoted by A_∞ and M_∞. Analogous formulas are valid for the components v_n and v_τ of the velocity vector $\mathbf{v}$ along the axes n and τ. We shall also mark the remaining values of the gas parameters in the incident undisturbed flow by the subscript ∞.

Proceeding from the results of papers ([1,3]), we introduce the following independent dimensionless variables and expansions for the sought functions:

$$\begin{gathered}
n = Ln', \qquad \tau = (L/\Delta)\tau', \\
v_n = a_\infty[1 + \varepsilon(v_{n1} + \delta v_{n2} + ...)], \\
v_\tau = a_\infty[\operatorname{ctg} A_\infty + \varepsilon\Delta(v_{\tau 1} + \delta v_{\tau 2} + ...)], \\
\rho = \rho_\infty[1 + \varepsilon(\rho_1 + \delta \rho_2 + ...)], \\
p = p_\infty[1 + \varepsilon(p_1 + \delta p_2 + ...)].
\end{gathered} \tag{2}$$

Here L denotes the characteristic dimension in the direction of the x-axis, and ε, Δ, δ are numerical parameters which are much smaller than unity in magnitude. Substitution of relations (2) into the system of equations describing the gas flow in the coordinates n and τ yields three dimensionless coefficients: the Reynolds numbers $\mathrm{Re}_1 = \rho_\infty a_\infty L/\lambda_1$, $\mathrm{Re}_2 = \rho_\infty a_\infty L/\lambda_2$, and the Péclet number $\mathrm{Pe} = \rho_\infty a_\infty c_p L/k$, whose reciprocals have the same order and are much smaller than unity.

In ([1]), from the continuity equation and the Navier–Stokes equations in projections on the axes n and τ, relations were derived connecting the functions of the first approximation*:

$$\rho_1 = \frac{p_\infty}{\rho_\infty a_\infty^2} p_1 = -v_{n1}, \qquad \frac{\partial v_{n1}}{\partial \tau} = \frac{\partial v_{\tau 1}}{\partial n}. \tag{3}$$

Using now the functional relations given by formulas (3), and retaining only the principal terms, we derive the equations of the second approximation from the continuity equation and the Navier–Stokes equations in projections on the axes n and τ, respectively,

$$\delta \frac{\partial}{\partial n}(\rho_2 + v_{n2}) = 2\varepsilon v_{n1} \frac{\partial v_{n1}}{\partial n} + \Delta \sqrt{M_\infty^2 - 1} \left(\frac{\partial v_{n1}}{\partial \tau} + \frac{v_{n1}}{\tau} \right) - \Delta^2 \left(\frac{\partial v_{\tau 1}}{\partial \tau} + \frac{v_{\tau 1}}{\tau} \right),$$

$$\delta \left(v_{n2} + \frac{p_\infty}{\rho_\infty a_\infty^2} p_2 \right) = -\Delta \sqrt{M_\infty^2 - 1}\, v_{\tau 1} + \frac{1}{\mathrm{Re}} \frac{\partial v_{n1}}{\partial n}, \qquad \frac{\partial v_{n2}}{\partial \tau} = \frac{\partial v_{\tau 2}}{\partial n}. \tag{4}$$

Here and in what follows Re denotes the total Reynolds number

$$\mathrm{Re}^{-1} = 4\mathrm{Re}_1^{-1}/3 + \mathrm{Re}_2^{-1},$$

which is associated with the so-called longitudinal viscosity.

From the third of equations (4) it is seen that the flow is irrotational not only in the first approximation, but also in the second. The influence of dissipative factors on the flow field will be obtained by considering the heat-transfer equation taken in the form ([1])

$$\rho \left[(v_n^2 - a^2) \frac{\partial v_n}{\partial n} + v_n v_\tau \left(\frac{\partial v_n}{\partial \tau} + \frac{\partial v_\tau}{\partial n} \right) + (v_\tau^2 - a^2) \frac{\partial v_\tau}{\partial \tau} - a^2 \frac{\sqrt{M_\infty^2 - 1}\, v_n - v_\tau}{\sqrt{M_\infty^2 - 1}\, n - \tau} \right] =$$

$$= v_n L_n(\lambda_1, \lambda_2) + v_\tau L_\tau(\lambda_1, \lambda_2) - \frac{\alpha a^2}{c_p} L(k, \lambda_1, \lambda_2). \tag{5}$$

Here $L_n(\lambda_1, \lambda_2)$ and $L_\tau(\lambda_1, \lambda_2)$ denote the right-hand sides of the n- and τ-components of the momentum-conservation equation without the terms $\partial p/\partial n$ and $\partial p/\partial \tau$, respectively, while $L(k, \lambda_1, \lambda_2)$ denotes the right-hand side of the heat-transfer equation.

The coefficients of equation (5) contain the speed of sound a. If the new dimensionless coefficients are denoted by m_1, m_2, m_3,

$$m_1 = \frac{1}{2\rho^3 a^2}\left(\frac{\partial^2 p}{\partial V^2}\right)_s, \qquad m_2 = \frac{1}{2\rho^4 a^2}\left(\frac{\partial^3 p}{\partial V^3}\right)_s, \qquad m_3 = \frac{c_p(\varkappa - 1)}{\alpha a T}\left(\frac{\partial a}{\partial s}\right)_p.$$

* Primes over dimensionless variables are omitted here and henceforth.

then the speed of sound, in the approximation needed by us, will be represented as

$$a = a_\infty \left\{1 - \varepsilon(m_{1\infty} - 1)v_{n1} - \frac{1}{2}\varepsilon^2(3m_{1\infty}^2 + m_{2\infty})v_{n1}^2 - \varepsilon\delta(m_{1\infty} - 1)v_{n2} + \frac{\varepsilon}{\mathrm{Re}}\left(m_{1\infty} - 1 - \frac{m_{3\infty}}{\mathrm{Pr}}\right)\frac{\partial v_{n1}}{\partial n} - \varepsilon\Delta \right. \tag{6}$$

The quantity $\mathrm{Pr} = \mathrm{Pe}\cdot\mathrm{Re}^{-1}$ is the Prandtl number. It is convenient also to use the following dimensionless thermodynamic parameters:

$$m_4 = \frac{\alpha a^2}{c_p}, \qquad m_5 = \frac{c_p(\varkappa - 1)^2}{\alpha^2 a^2}\left(\frac{\partial}{\partial T}\frac{\alpha a^2}{c_p}\right)_s, \qquad \mu_1 = \frac{\varkappa - 1}{\alpha}\,\frac{{}^4/_3\, d\lambda_1/dT + d\lambda_2/dT}{{}^4/_3\lambda_1 + \lambda_2},$$

$$\mu_2 = \frac{\rho(\varkappa - 1)}{k}\left(\frac{\partial}{\partial T}\frac{k}{\alpha\rho}\right)_s, \qquad \mu_3 = \frac{\rho a^2(\varkappa - 1)}{k}\left(\frac{\partial}{\partial T}\frac{\varkappa k}{\alpha\rho a^2}\right)_s.$$

Let us now simplify equation (5). For this purpose (3), (4), and (6) are used. We give the result of the transformation immediately in its final form:

$$
\begin{aligned}
&2\varepsilon m_{1\infty} v_{n1}\frac{\partial v_{n1}}{\partial n} + \Delta\sqrt{M_\infty^2-1}\left(2\frac{\partial v_{n1}}{\partial \tau} + \frac{v_{n1}}{\tau}\right) - \frac{1}{\mathrm{Re}}\left(1+\frac{\varkappa-1}{\mathrm{Pr}}\right)\frac{\partial^2 v_{n1}}{\partial n^2}\\
&\quad + 2\varepsilon\delta m_{1\infty}\left(v_{n1}\frac{\partial v_{n2}}{\partial n} + v_{n2}\frac{\partial v_{n1}}{\partial n}\right) + \varepsilon^2(2m_{1\infty}^2 + m_{2\infty})v_{n1}^2\frac{\partial v_{n1}}{\partial n}\\
&\quad + \varepsilon\Delta\sqrt{M_\infty^2-1}\left[2(m_{1\infty}-1)v_{\tau1}\frac{\partial v_{n1}}{\partial n} - (2m_{1\infty}-1)\frac{v_{n1}^2}{\tau}\right]\\
&\quad - \Delta^2\left[(2-M_\infty^2)\frac{\partial v_{\tau1}}{\partial \tau} + \frac{v_{\tau1}}{\tau} - (M_\infty^2-1)\frac{nv_{n1}}{\tau^2}\right] + \delta\Delta\sqrt{M_\infty^2-1}\left(2\frac{\partial v_{n2}}{\partial\tau} + \frac{v_{n2}}{\tau}\right)\\
&\quad - \delta\Delta^2\left[(2-M_\infty^2)\frac{\partial v_{\tau2}}{\partial\tau} + \frac{v_{\tau2}}{\tau}\right] + \varepsilon\Delta^2\left[2v_{\tau1}\frac{\partial v_{n1}}{\partial\tau} + v_{n1}\frac{\partial v_{\tau1}}{\partial\tau}(2m_{1\infty} - M_\infty^2)\right.\\
&\qquad \left. +(2m_{1\infty}-1)\frac{v_{n1}v_{\tau1}}{\tau}\right] - \frac{\varepsilon}{\mathrm{Re}}\left[\left(1-\mu_{1\infty}+\frac{\nu_{1\infty}}{\mathrm{Pr}}\right)v_{n1}\frac{\partial^2 v_{n1}}{\partial n^2} - \left(\nu_{2\infty} - \frac{\nu_{3\infty}}{\mathrm{Pr}}\right)\left(\frac{\partial v_{n1}}{\partial n}\right)^2\right]\\
&\quad - \frac{\delta}{\mathrm{Re}}\left(1+\frac{\varkappa_\infty-1}{\mathrm{Pr}}\right)\frac{\partial^2 v_{n2}}{\partial n^2} - \frac{\Delta^2}{\mathrm{Re}}\left(1+\frac{\varkappa_\infty-2}{\mathrm{Pr}}\right)\frac{\partial}{\partial n}\left(\frac{\partial v_{\tau1}}{\partial\tau} + \frac{v_{\tau1}}{\tau}\right)\\
&\quad + \frac{1}{\mathrm{Re}^2}\frac{\varkappa_\infty}{\mathrm{Pr}}\frac{\partial^3 v_{n1}}{\partial n^3} - \frac{\Delta}{\mathrm{Re}}\sqrt{M_\infty^2-1}\left(1+\frac{\varkappa_\infty-1}{\mathrm{Pr}}\right)\frac{\partial}{\partial n}\left(\frac{\partial v_{n1}}{\partial\tau} - \frac{v_{n1}}{\tau}\right)\\
&\quad + \frac{\Delta}{\mathrm{Re}\cdot\mathrm{Pr}}\sqrt{M_\infty^2-1}\,\frac{\partial}{\partial n}\left(2\frac{\partial v_{n1}}{\partial\tau} + \frac{v_{n1}}{\tau}\right) = 0.
\end{aligned}
\tag{7}
$$

To shorten the notation, the designations $\nu_1 = 2+\mu_2-\mu_3-m_5$, $\nu_2 = 2+\mu_1+m_4-2m_1$, $\nu_3 = 2+\mu_2-\mu_3-2m_3$ have been introduced.

The coefficients of equation (7) include the small parameters ε, Δ, δ, Re^{-1}, and the number M_∞; its subsequent simplification will depend on the choice of relative magnitudes of these parameters, while the flow regimes of the gas can be classified in different ways according to the magnitude of M_∞.

If $M_\infty = 1$, then equation (7) gives the equations of the first and second approximations and, together with the inviscid equations (3), (4), will describe the asymptotic picture of the flow of a real gas past bodies of revolution in a sonic stream ([3,4]).

Let now $M_\infty > 1$. From the results of works ([1,2]) it follows that the structure of the flow at infinity is determined mainly by dissipative factors, and we assume that

$$
\varepsilon \ll \Delta = \frac{1}{2}\,\mathrm{Re}^{-1}(M_\infty^2-1)^{-1/2}\left(1+\frac{\varkappa_\infty-1}{\mathrm{Pr}}\right).
$$

The equations of the first approximation were obtained in work (1)

$$
\partial^2 v_{n1}/\partial n^2 - \partial v_{n1}/\partial\tau - v_{n1}/2\tau = 0, \qquad \partial v_{\tau1}/\partial n = \partial v_{n1}/\partial\tau. \tag{8}
$$

The solution of the first of them is found directly from (2)

$$v_{n1} = Hnt^{-2}e^{-n^2/4\tau} \qquad (H = \text{const}). \tag{9}$$

Putting the parameter $\delta = \mathrm{Re}^{-1}(M_\infty^2 - 1)^{-1}\left(1 + \dfrac{\chi_\infty - 1}{\mathrm{Pr}}\right)/4$ and using relations (8), (9), we derive the equation of the second approximation

$$\frac{\partial^2 v_{n2}}{\partial n^2} - \frac{\partial v_{n2}}{\partial \tau} - \frac{v_{n2}}{2\tau} = 2(M_\infty^2 - 1)\left(1 + \frac{\chi_\infty + 1}{\mathrm{Pr}}\right)\left(1 + \frac{\chi_\infty - 1}{\mathrm{Pr}}\right)^{-1}\frac{\partial}{\partial n}\left(\frac{\partial v_{n1}}{\partial \tau} - \frac{v_{n1}}{\tau}\right) +$$

$$+ 4\mathrm{Pr}^{-1}(M_\infty^2 - 1)\left(1 + \chi_\infty + \frac{\chi_\infty - 1}{\mathrm{Pr}}\right)\left(1 + \frac{\chi_\infty - 1}{\mathrm{Pr}}\right)^{-2}\frac{\partial^3 v_{n1}}{\partial n^3} -$$

$$- \left[(2 - M_\infty^2)\frac{\partial v_{\tau 1}}{\partial \tau} + \frac{v_{\tau 1}}{\tau} - (M_\infty^2 - 1)\frac{n v_{n1}}{\tau^2}\right]. \tag{10}$$

As in the first approximation (2), the functions of the second approximation will be sought in self-similar form $v_{n2} = \tau^{-2} f(\xi)$, where $\xi = \frac{1}{2} n\tau^{-1/2}$. To determine $f(\xi)$ we obtain the differential equation

$$d^2 f/d\xi^2 + 2\xi\, df/d\xi + 6f = 4He^{-\xi^2}(\beta_1\xi^4 + \beta_2\xi^2 + \beta_3). \tag{11}$$

The introduced dimensionless parameters $\beta_1, \beta_2, \beta_3$ are negative outside the transonic range of velocities and have the form

$$\beta_1 = 2\left[2 - M_\infty^2 - 2(M_\infty^2 - 1)\left(1 + \frac{\chi_\infty - 1}{\mathrm{Pr}}\right)^{-2}\left(1 + \frac{2 + 4\chi_\infty}{\mathrm{Pr}} + \frac{\chi_\infty^2 + 2\chi_\infty - 3}{\mathrm{Pr}^2}\right)\right],$$

$$\beta_2 = 6\left[(M_\infty^2 - 1)\left(1 + \frac{\chi_\infty - 1}{\mathrm{Pr}}\right)^{-2}\left(3 + \frac{4 + 10\chi_\infty}{\mathrm{Pr}} + \frac{3\chi_\infty^2 + 4\chi_\infty - 7}{\mathrm{Pr}^2}\right) - 3 + 2M_\infty^2\right],$$

$$\beta_3 = 2\left[3 - 2M_\infty^2 - 3(M_\infty^2 - 1)\left(1 + \frac{\chi_\infty - 1}{\mathrm{Pr}}\right)^{-2}\left(1 + \frac{1 + 3\chi_\infty}{\mathrm{Pr}} + \frac{\chi_\infty^2 + \chi_\infty - 2}{\mathrm{Pr}^2}\right)\right].$$

The homogeneous equation (11) has the fundamental solutions $(1 - 2\xi^2)e^{-\xi^2}$ and $\xi\Phi(-1/2,\, 3/2;\, \xi^2)e^{-\xi^2}$, where Φ denotes the degenerate hypergeometric function. The second of them does not suit the problem posed, since in the region $\xi \to \pm\infty$ it gives us a function $v_{n2} \approx 2n^{-3}\tau^{-1/2}$, satisfying the acoustic equation (2). Hence the general solution of the inhomogeneous equation (11), obtained by the method of variation of constants, has the form

$$f(\xi) = \Big[N(1-2\xi^2) + 4H\xi\Phi(-1/2,\,3/2;\,\xi^2)\int(1-2\xi^2)(\beta_1\xi^4+\beta_2\xi^2+\beta_3)e^{-\xi^2}\,d\xi$$

$$-4H(1-2\xi^2)\int\xi(\beta_1\xi^4+\beta_2\xi^2+\beta_3)\Phi(-1/2,\,3/2;\,\xi^2)e^{-\xi^2}\,d\xi\Big]\,e^{-\xi^2} \qquad (N = \text{const}).$$

With the help of the solution found it is easy to obtain the asymptotic behavior of the function $f(\xi)$ as $\xi \to \pm\infty$, namely:

$$f(\xi) = -\left[\beta_1 H\xi^4 + 2\beta_2\xi^2\ln|\xi| + (2N+\beta_1 H)\xi^2 + ...\right]e^{-\xi^2}.$$

Let us now consider the dependences that exist between the small parameters $\varepsilon, \delta, \Delta, \mathrm{Re}^{-1}$ in the case $M_\infty > 1$. The dependence between the quantities $\delta, \Delta, \mathrm{Re}^{-1}$ was already given above. The convective term does not enter equation (10), since it begins to affect the flow field in the next approximation, and it is not possible to establish directly the relation between ε and the remaining parameters $\delta, \Delta, \mathrm{Re}^{-1}$. However, using the self-similarity of the solutions and equation (7), it can be derived. We have

$$\mathrm{Re}^{-1} \sim \varepsilon^{1/3}(M_\infty^2-1)^{2/3} \approx \Delta(M_\infty^2-1)^{1/2} \sim \delta(M_\infty^2-1).$$

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Note: Figure translations are in progress. See original paper for figures.