THE NUMBER, DEGREE OF DISTINGUISHABILITY, AND DIAMETER OF PERMUTATION AUTOMATA AND OF THE OPERATORS REALIZED BY THEM

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Abstract

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A. D. KORSHUNOV

THE NUMBER, DEGREE OF DISTINGUISHABILITY, AND DIAMETER OF PERMUTATION AUTOMATA AND OF THE OPERATORS REALIZED BY THEM

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1°. In the present article, by an automaton we shall everywhere mean a Mealy automaton in which all states are pairwise distinguishable* and one of them is declared initial. We note that in such an automaton not every state need be reachable from the initial one. For any m, n, and k, let $R(m, n, k)$ denote the set of all automata with input letters $x_1, x_2, \ldots, x_m$, output letters $y_1, y_2, \ldots, y_n$, and states $q_1, q_2, \ldots, q_k$. Let $T(m, n, k)$ be the subset of the set $R(m, n, k)$ consisting of connected automata, i.e., automata in which every state is reachable from the initial state. If in $T(m, n, k)$ we do not distinguish automata isomorphic (with respect to states), then each of them can be identified with the operator realized by it. Let $h_A(q_i, q_j)$ be the degree of distinguishability of the states q_i and q_j in an automaton A from $R(m, n, k)$. The quantity

$$h_A = \max h_A(q_i, q_j),$$

where the maximum is taken over all pairs of states in A, is called the **degree of distinguishability** of the automaton A. Let $d_A(q_i, q_j)$ be the length of the shortest input word taking $A \in R(m, n, k)$ from q_i to q_j. The **diameter** of the automaton A is the quantity

$$d_A = \max d_A(q_i, q_j),$$

where the maximum is taken over all pairs (q_i, q_j) such that q_j is reachable from q_i in A.

By the degree of distinguishability and the diameter of an operator is meant the degree of distinguishability and the diameter of a connected automaton realizing this operator.

The degree of distinguishability and the diameter are the most important parameters of automata and of the operators realized by them, and estimating their magnitude is of great importance in the formulation and solution of many problems in abstract automata theory: synthesis, minimization, experiments, etc. For different automata with one and the same number of states, the values of each of the named parameters are, generally speaking, different. Even at the beginning of the development of automata theory it was established ([2], [3]) that the maxima of the degree of distinguishability and of the diameter of an automaton are less by one than the number of its states. However, the question of the most probable values of these parameters long remained open. It was solved only after in ([4]) an asymptotic estimate was found for the cardinality of the set $R(m, n, k)$. Namely, it was proved that the degree of distinguishability ([5]) and the diameter ([6]) of almost all automata from $R(m, n, k)$ are substantially smaller than the maximal possible ones.

We note that the cardinality of the set $T(m, n, k)$ is negligible in comparison with the cardinality of the set $R(m, n, k)$. Therefore the estimates for the degree of distinguishability and the diameter obtained for almost all automata from $R(m, n, k)$ do not transfer directly to almost all automata from $T(m, n, k)$.

Meanwhile, in the theory of automata synthesis such estimates are important precisely in the class $T(m, n, k)$. Consider, for example, the problem of restoring an operator, arising in the theory of synthesis and in the theory of experiments. Let,

* Concepts not defined here can be found, for example, in ([1]).

the "customer" has in mind some operator or, what is the same thing, some automaton A from $T(m, n, k)$. The "performer," to whom the parameters m, n, and k are known, must guess the automaton A that has been conceived, and in doing so has the right to demand from the customer answers to questions of the type "into what does A transform such-and-such an input word?"

It is known ([1]) that if the automaton A that has been conceived has degree of distinguishability h and diameter α, then A is reconstructed if it is established how it transforms input words of length $h+\alpha+1$ into output words of the same length, i.e., A is reconstructed by means of a multiple experiment of length $h + \alpha + 1$.

However, in the class $T(m, n, k)$ the most probable values of the degree of distinguishability and the diameter are still unknown.

In connection with this circumstance, the problem arises of singling out in $R(m, n, k)$ such "natural" subclasses for which: a) almost all automata of the subclass under consideration are minimal; b) it is possible to establish the most probable values of the degree of distinguishability and the diameter. This would make it possible to transfer the estimates obtained to the corresponding classes of operators.

The purpose of the present paper is precisely to study one special class of automata, the so-called permutation automata. A characteristic feature of this class is that many results obtained for almost all automata of this class carry over to almost all permutation operators. For this class of automata we are able to obtain results analogous (and in some cases more definitive) to those established in ([4–6]). Before proceeding to the formulation of these results, let us introduce some concepts.

A connected automaton is called **strongly connected** if its transition diagram is a strongly connected graph ([7]). An automaton is called **permutation** if any two distinct states, under the action of each input letter, pass into distinct states ([8]). An operator is called **permutation** if it is realized in a minimal permutation automaton.

2°. Let $A_{\text{per}}(m, n, k)$ and $A^{\text{sc}}_{\text{per}}(m, n, k)$ be, respectively, the set of permutation automata from $R(m, n, k)$ and the set of permutation automata from $T(m, n, k)$. We are interested in the behavior of these and certain other quantities when m, n, and k satisfy certain restrictions. More precisely, we consider a family of triples $\{m_1, n_1, k_1\}$, $\{m_2, n_2, k_2\}, \ldots$ and are interested in the behavior of the parameters under study for automata from $A_{\text{per}}(m_i, n_i, k_i)$ and the operators realized by them, when the sequence of triples $\{m_i, n_i, k_i\}_{i=1}^{\infty}$ satisfies certain restrictions. In particular, throughout what follows it is assumed that $m_i \geq 2$ and $n_i \geq 2$ for every i, and we shall not stipulate this separately each time. We now proceed to the formulation of the results.

Theorem 1. If $m_i + n_i + k_i \to \infty$ as $i \to \infty$, then*

$$\left|A_{\text{per}}(m_i, n_i, k_i)\right| \sim k_i \left(k_i!\, n_i^{k_i}\right)^{m_i}.$$

Corollary 1. The number of pairwise nonisomorphic (with respect to states) automata from $A_{\text{per}}(m_i, n_i, k_i)$ is asymptotically equal to

$$\left(k_i!\, n_i^{k_i}\right)^{m_i} / (k_i - 1)!, \quad \text{if } m_i + n_i + k_i \to \infty \text{ as } i \to \infty.$$

The validity of this assertion follows from the fact that, as a result of all possible renumberings of the states of any automaton from $R(m, n, k)$, exactly $k!$ distinct automata are obtained.

Theorem 2. If $m_i + k_i \to \infty$ as $i \to \infty$, then

$$\left|A^{\text{sc}}_{\text{per}}(m_i, n_i, k_i)\right| \sim \left|A_{\text{per}}(m_i, n_i, k_i)\right|.$$

* By $|M|$ we denote the cardinality of the set M.

Remark 1. This result is analogous to the result that almost all automata from $R(m_i, n_i, k_i)$ are connected ([4]), if $m_i + n_i \to \infty$ as $i \to \infty$.

Denote by $A^{\text{s. conn}}_{\text{per}}(m, n, k)$ the subset of the set $A^{\text{conn}}_{\text{per}}(m, n, k)$ consisting of strongly connected automata.

Corollary 2. If $m_i + k_i \to \infty$ as $i \to \infty$, then

$$\left|A_{\text{per}}^{\text{s. conn}}(m_i, n_i, k_i)\right| \sim \left|A_{\text{per}}(m_i, n_i, k_i)\right|.$$

The validity of this assertion follows from Theorem 2 and from the easily verified fact that every permutation automaton is a strongly connected automaton.

Let $T_{\text{per}}(m, n, k)$ be the set of pairwise distinct operators realized by automata from $A_{\text{per}}^{\text{conn}}(m, n, k)$.

Corollary 3. If $m_i + k_i \to \infty$ as $i \to \infty$, then

$$\left|T_{\text{per}}(m_i, n_i, k_i)\right| = \left|A_{\text{per}}^{\text{s. conn}}(m_i, n_i, k_i)\right| / k_i! \sim (k_i!\, n_i^{k_i})^{m_i} / (k_i - 1)!.$$

3°. Let $\{M(r)\}_{r=1}^{\infty}$ be a family of sets depending on a parameter r such that $|M(r)| \to \infty$ as $r \to \infty$, and let $M_E(r)$ be the subset of the set $M(r)$ each element of which has the prescribed property E. It is said that almost all elements of the set $M(r)$ have property E if $|M_E(r)|/|M(r)| \to 1$ as $r \to \infty$.

Theorem 3. If $m_i = \text{const}$, $n_i = \text{const}$, and $k_i \to \infty$ as $i \to \infty$, then for almost all automata A from $A_{\text{per}}(m_i, n_i, k_i)$ the degree of distinguishability h_A satisfies the relation

$$[\log_{m_i} \log_{n_i} k_i] \le h_A \le [\log_{m_i} \log_{n_i} k_i] + 4.$$

Remark 2. The estimates given in this theorem coincide with analogous estimates for the degree of distinguishability of almost all automata from $R(m, n, k)$ ([5]).

Corollary 4. If $m_i = \text{const}$, $n_i = \text{const}$, and $k_i \to \infty$ as $i \to \infty$, then for almost all permutation operators T, realizable by automata from $A_{\text{per}}^{\text{s. conn}}(m_i, n_i, k_i)$, the degree of distinguishability h_T satisfies the relation

$$[\log_{m_i} \log_{n_i} k_i] \le h_T \le [\log_{m_i} \log_{n_i} k_i] + 4.$$

Theorem 4. If $m_i = \text{const}$, $n_i = \text{const}$, and $k_i \to \infty$ as $i \to \infty$, then for almost all automata A from $A_{\text{per}}(m_i, n_i, k_i)$ the diameter d_A satisfies the relation

$$d_A \sim \log_{m_i} k_i.$$

Remark 3. This result is more definitive in comparison with the result of ([6]), in which for almost all automata from $R(m, n, k)$ lower and upper estimates of the diameter were established, differing from one another by a multiplicative constant.

Corollary 5. If $m_i = \text{const}$, $n_i = \text{const}$, and $k_i \to \infty$ as $i \to \infty$, then for almost all permutation operators T, realizable by automata from $A^{\text{s. conn}}_{\text{per}}(m_i, n_i, k_i)$, the diameter d_T satisfies the relation

$$d_T \sim \log_{m_i} k_i.$$

Corollary 6. If $m_i = \text{const}$, $n_i = \text{const}$, and $k_i \to \infty$ as $i \to \infty$, then almost all permutation operators realized by automata from $A^{\text{s. conn}}_{\text{per}}(m_i, n_i, k_i)$ are reconstructed by means of multiple experiments of length

$$l \sim \log_m k_i.$$

4°. An automaton is called a Moore automaton if its output is a function of a single variable—the internal state. Using the method of proof of Theorems 1-4, one can verify that for pere-

of Moore automata and of the operators realized by them there are analogous assertions.

The main ideas of the proofs of Theorems 1-4 are borrowed from ([4–6]). The author expresses his gratitude to B. A. Trakhtenbrot for valuable advice given during the writing of this work.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.