ASSOCIATED NUCLEAR TOPOLOGY, MAPPINGS OF TYPE s, AND STRONGLY NUCLEAR SPACES

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Abstract

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ASSOCIATED NUCLEAR TOPOLOGY, MAPPINGS OF TYPE s, AND STRONGLY NUCLEAR SPACES

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Grothendieck, in his monograph ([2]), proposed classifying nuclear spaces by the properties of "decrease" of equicontinuous subsets of their conjugates. In the present paper a number of questions closely connected with the "decrease" of equicontinuous subsets are considered.

1. Definition 1. Let E be a locally convex space and E' its conjugate. A sequence $(x_n)_{n\in N}$ in E is called **rapidly decreasing** if $n^k x_n \to 0$ in E for every $k \in N$. A set $A \subset E$ will be called **nuclear** if A is contained in the closed absolutely convex hull of some rapidly decreasing sequence in E. A sequence $(x'_n)_{n\in N}$ in E' will be called **rapidly decreasing** if the set $\{n^k x'_n\}_{n\in N}$ is equicontinuous in E' for any $k \in N$; a set $B \subset E'$ will be called **nuclear** if B is contained in the $\sigma(E', E)$-closed absolutely convex hull of some rapidly decreasing sequence in E'.

From the results of Grothendieck (([2]), Ch. II, p. 67) and T. and Y. Komura ([4]) there follows a criterion for nuclearity of a locally convex space.

Theorem 1 (T. and Y. Komura, Grothendieck). *In order that a locally convex space E be nuclear, it is necessary and sufficient that every equicontinuous set in E' be nuclear.*

Definition 2. Let (E, t) be a locally convex space. The strongest nuclear topology t_N in E majorized by the topology t will be called the **nuclear topology associated with** t.

Using Theorem 1, we obtain:

Theorem 2. *The nuclear topology t_N associated with the topology t of the locally convex space (E, t) is the topology of uniform convergence on all nuclear sets from E'.*

Theorem 1 follows easily from Theorem 2. There is a proof of Theorem 2 that does not rely on the apparatus of tensor products (used by Grothendieck).

2. Let E and F be normed spaces, $f \in L(E,F)$,

$$A_n(E,F) = \{g \in L(E,F) : \dim g(E) \leq n\}$$

and

$$\alpha_n(f) = \inf\{\|f-g\| : \ g \in A_n(E,F)\}.$$

Definition 3 (Pietsch). f is called a **mapping of type** s,* if

$$\sum_{n\in N} (\alpha_n(f))^p < \infty$$

for every $p > 0$.

Theorem 3. *For a mapping $f \in L(E,F)$ the following conditions are equivalent:*

a) f is a mapping of type s,

b) the image of the unit ball of E is a nuclear set in F,

c) $f : (E, t_N) \to F$ is continuous,

d) in E' there exists a rapidly decreasing sequence $(x'_n)_{n\in N}$ such that

$$\|f(x)\| \leq \sum |\langle x, x'_n\rangle|$$

for every $x \in E$.

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* Mappings of type s coincide with Fredholm mappings of zero order introduced by Grothendieck ([2]).

From one theorem of Pietsch ((^5), 8.5.6) it follows that the adjoint of a mapping of type s is also a mapping of type s. From Theorem 3 it follows that the converse is also true.

Proposition 1. Let $f \in L(E,F)$. If the adjoint of f is a mapping of type s, then f itself is also a mapping of type s.

3. For each neighborhood of zero U of the locally convex space E, denote by E'_{U^0} the vector subspace in E' spanned by the polar U^0 of the neighborhood U, with the normed topology determined by the unit ball U^0.

Definition 4. A locally convex space E will be called **strongly nuclear** if, for every neighborhood of zero U in E, there exists a neighborhood of zero V such that the canonical mapping $E'_{U^0} \to E'_{V^0}$ is a mapping of type s.*

The class of strongly nuclear spaces includes: the space $A(G)$ of all analytic functions on an open set G of the complex plane, the space D' of distributions of L. Schwartz, the strong duals of metrizable nuclear spaces, and others. The class of strongly nuclear spaces is stable under the same operations as the class of nuclear spaces. Namely:

Proposition 2. Subspaces and products, and consequently projective limits, of strongly nuclear spaces are strongly nuclear.

Proposition 3. A quotient space and a direct sum of a sequence, and consequently an inductive limit of a sequence, of strongly nuclear spaces are strongly nuclear spaces.

4. We shall give a characterization of strongly nuclear spaces in terms of the "smallness" of equicontinuous subsets of their duals.

Definition 5. Let E be a locally convex space. A sequence $(x'_n)_{n\in\mathbb{N}}$ from E' will be called **locally rapidly decreasing** if $(x'_n)_{n\in\mathbb{N}}$ is rapidly decreasing in E'_{V^0} for some neighborhood of zero V in E; a set $B \subset E'$ will be called **strongly nuclear** if B is contained in the $\sigma(E', E)$-closed absolutely convex hull of some locally rapidly decreasing sequence from E'.

Theorem 4. Let (E, t) be a locally convex space. Then the topology t_{sN} of uniform convergence on all strongly nuclear subsets of E' is the strongest strongly nuclear topology in E majorized by the topology t.

Corollary. In order that a locally convex space E be strongly nuclear, it is necessary and sufficient that every equicontinuous set in E' be strongly nuclear.

Let t_N be the nuclear topology associated with the normed topology of the Hilbert space l^2.

Theorem 5. A separable locally convex space is strongly nuclear if and only if it is isomorphic to a subspace of the space $(l^2, t_N)^I$ (= the product of I copies of the space (l^2, t_N)).

Let LC be the category** of locally convex spaces, and SN the category of strongly nuclear spaces with their continuous linear mappings. The proof of Theorem 5 is based on the fact that the correspondence $S : LC \to SN$ such that $S(E, t) = (E, t_{sN})$ and $S(f) = f : (E, t_{sN}) \to (F, t'_{sN})$ for $f : (E, t) \to (F, t')$, is a functor having a left adjoint.

5. Let E be a locally convex space and $\tilde{E}$ its local completion (ˆ6). The following theorem (cf. with the theorems of Pietsch (ˆ5), 4.3.1 and 9.5.2) gives a condition for the nuclearity of the strong dual of a locally convex space.

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* **Note added in proof.** After the note had been submitted to the editors, the author learned that strongly nuclear spaces had previously been studied by A. Martineau. He also proved Propositions 2, 3 and the corollary to Theorem 4 of the present note (see (ˆ7)).

** For the notions of a category, a functor, and an adjoint functor, see (ˆ3).

Theorem 6. In order that the strong dual of E be nuclear, it is necessary and sufficient that every bounded set in E be nuclear in $\tilde{E}$.

Theorem 7. In order that the strong dual of E be strongly nuclear, it is necessary and sufficient that for every bounded set A in E there exist a bounded set B in E such that A is contained and nuclear in E_B.

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Note: Figure translations are in progress. See original paper for figures.