ON THE PRESERVATION OF THE MULTIPLICITY OF A BASE UNDER A PERFECT MAPPING

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ON THE PRESERVATION OF THE MULTIPLICITY OF A BASE UNDER A PERFECT MAPPING

(Presented by Academician P. S. Aleksandrov on XII 1, 1967)

In the present note a positive answer is given to the question posed by P. S. Aleksandrov: is the multiplicity of a base preserved under a perfect mapping?

Definition*. We say that the **multiplicity of a family Φ of subsets of a set F does not exceed the cardinal number** τ, if each point of the set F is contained in no more than τ elements of the family Φ.

In fact it turned out that even the following is true.

Theorem 1. *Let $f : X \to Y$ be a pseudo-open bicompact mapping of a T_1* ***space*** *X**, possessing a base of multiplicity** $\leq \tau$**, onto the space** Y**. Then** Y ***also possesses a base of multiplicity*** $\leq \tau$.*

Corollary. *Let $f : X \to Y$ be a perfect mapping, and let X have a base of multiplicity $\leq \tau$. Then Y also has a base of multiplicity $\leq \tau$.*

This is precisely the answer to P. S. Aleksandrov' s question mentioned above.

In the present note we shall give a direct proof of the corollary and an outline of the proof of Theorem 1.

We begin the proof of the corollary with the following lemma.

Lemma. *Let $Z' \subseteq Z$ be arbitrary sets, and let θ', θ be families of subsets, respectively of Z' and Z, whose multiplicity does not exceed τ. Consider the sets*

$$\gamma_{\vartheta'} = \bigcup\{\vartheta,\ \vartheta \in \theta,\ \emptyset \neq \vartheta \cap Z' \subseteq \vartheta'\}.$$

Under the assumptions made, the multiplicity of the family $\Gamma = \{\gamma_{\vartheta'}\}_{\vartheta' \in \theta'}$ does not exceed τ.

First of all, the set $\vartheta \cap Z'$ cannot be contained in more than τ elements of the family θ', since the multiplicity of θ' is not greater than τ. Therefore $\vartheta \in \theta$

occurs as a summand in no more than τ sets $\gamma_{\vartheta'}$. A point $z \in Z$ is contained only in those elements $\gamma_{\vartheta'} \in \Gamma$ for which there is a $\vartheta \in \theta$, contained in $\gamma_{\vartheta'} \in \Gamma$ as a summand and containing z; and, by what has been said above, there are no more than $\tau^2 = \tau$ such elements, as was required.

Let B^X be a fixed base of the space X, whose multiplicity is $\leq \tau$, and let B_n^X $(n = 1, 2, ...)$ be the set of families of elements of the base B^X

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* All other definitions, as well as the necessary comments, may be found in (1).

** Here, and everywhere below, although this is not specially stated, we require that the image space be a T_1-space, so that A. S. Mishchenko' s theorem (2) may be used. All mappings are assumed to be mappings onto.

*** *Note added in proof.* After submitting the article to the editors the author obtained a stronger result:

Theorem. *Let $f : X \to Y$ be a bifactorial τ-mapping of a T_1-space X, possessing a base of multiplicity $\leq \tau$, onto a space Y. Then Y also possesses a base of multiplicity $\leq \tau$.*

A mapping $f : X \to Y$ is called **bifactorial** if from every cover $\{U_\alpha\}$ of the preimage $f^{-1}(y)$ of any point $y \in Y$ one can choose a finite number of elements $\{U_1, ..., U_n\}$ such that $y \in \operatorname{Int} f(U_1 \cup \cdots \cup U_n)$, and it is called a τ-mapping if the weight of the preimage of any point $y \in Y$ does not exceed τ.

$\beta = \{b_1, ..., b_n\}$, for which there exists a point $y \in Y$ such that the sets $b_1, ..., b_n$ form a minimal covering* of its inverse image $f^{-1}(y)$. The inverse image of each such point $y \in Y$ we shall call an **essential element** of the family β. By induction on n and on all possible triples $f : X \to Y$, where the space X has a base B^X of multiplicity $\leqslant \tau$, and the mapping f is perfect, we shall prove that there exists a family Φ_n^X of multiplicity $\leqslant \tau$ of open subsets of the space X and a one-to-one correspondence $g_n^X : B_n^X \to \Phi_n^X$ such that

$$g_n^X(\beta) \subseteq \bigcup_{b \in \beta} b$$

and $g_n^X(\beta)$ contains all essential elements of β. (Here, incidentally, a situation may arise in which two elements $\varphi_1, \varphi_2 \in \Phi_n^X$ coincide as sets lying in the space X, but, being put into correspondence with different elements of the family B_n^X, they will differ as elements of Φ_n^X.)

As Φ_1^X we take B_1^X and as g_1^X the identity mapping. The multiplicity of $B_1^X \leqslant \tau$, since $B_1^X \subseteq B^X$ and the multiplicity of $B^X \leqslant \tau$.

Suppose the assertion has been proved for any space X and $n = k$. We prove the assertion for $n = k + 1$. Take an arbitrary $b \in B^X$. Consider $X' = X \setminus b$. As $B^{X'}$ we take the intersections of the elements of B^X with X'. By the induction hypothesis** for X' there has been constructed a family $\Phi_k^{X'}$ and a correspondence $g_k^{X'}$. To each set $\varphi' \in \Phi_k^{X'}$ assign the set

$$\varphi = f^{-1}f(X') \setminus f^{-1}f(X' \setminus \varphi').$$

We apply the lemma, putting $\theta' = \{\varphi\}$, $\theta = B^X$, $Z' = f^{-1}f(X')$, $Z = X$. From A. S. Mishchenko' s theorem ([2]) it follows easily that the multiplicity of the family $\{\varphi\}$ does not exceed τ, and hence we are in the conditions of the lemma. The family $\{\gamma_\varphi\} = \Gamma_{k+1}^b$ constructed in accordance with the lemma has multiplicity $\leqslant \tau$. As a sum of open sets, each element Γ_{k+1}^b is open. Let Δ_{k+1}^b be the part of B_{k+1}^X consisting of the families containing the set b. Define the correspondence

$$g_{k+1}^b : \Delta_{k+1}^b \to \Gamma_{k+1}^b$$

as follows:

$$g_{k+1}^b(\{b, b_1, \dots, b_k\}) = \gamma_{f^{-1}f(X') \setminus f^{-1}f\left(X' \setminus g_k^{X'}(\{b_1 \cap X', \dots, b_k \cap X'\})\right)}.$$

Let us note some properties of the correspondence g_{k+1}^b. $g_{k+1}^b(\beta) \subseteq \bigcup_{b\in\beta} b$—this follows from the inclusions

$$\varphi = f^{-1}f(X') \setminus f^{-1}f(X' \setminus \varphi') \subseteq b \cup \varphi',$$

where

$$\varphi' = g_k^{X'}(\{b_1 \cap X', \dots, b_k \cap X'\}) \subseteq \bigcup_{i=1}^{k}(b_i \cap X') \subseteq \bigcup_{i=1}^{k} b_i,$$

$$\gamma_\varphi \subseteq (X \setminus f^{-1}f(X')) \cup \varphi \subseteq b \cup \varphi \subseteq b \cup \varphi' \subseteq b \cup b_1 \cup \dots \cup b_k.$$

We show that $g_{k+1}^b(\beta)$ contains all essential elements of β. If $y \in Y$ and $f^{-1}(y)$ is an essential element of $\{b, b_1, \dots, b_k\}$, then $f^{-1}(y) \setminus b$, evidently, is an essential element of $\{b_1 \cap X', \dots, b_k \cap X'\}$ (for $f|_{X'}$, the restriction of the mapping f to X'), and therefore

$$f^{-1}(y) \setminus b \subseteq g_k^X(\{b_1 \cap X', \dots, b_k \cap X'\}).$$

The set

$$X' \setminus g_k^X(\{b_1 \cap X', \dots, b_k \cap X'\})$$

is closed in X, and therefore the set

$$f^{-1}f(X' \setminus g_k^X(\{b_1 \cap X', \dots, b_k \cap X'\}))$$

is also closed in X and does not meet $f^{-1}(y)$. For each point $x \in f^{-1}(y)$ there is found a base element not meeting the set

$$f^{-1}f(X' \setminus g_k^{X'}(\{b_1 \cap X', \dots, b_k \cap X'\})).$$

All such elements will enter into

$$\gamma_{f^{-1}f(X')\setminus f^{-1}f\left(X'\setminus g_k^{X'}(\{b_1\cap X',\dots,b_k\cap X'\})\right)}$$

as summands, whence it follows that

$$f^{-1}(y)\subseteq g^b_{k+1}(\{b,b_1,\dots,b_k\}).$$

We construct the family Φ^X_{k+1} in the following way. Let $\{b_1,\dots,b_{k+1}\}\in B^X_{k+1}$; then

$$g^X_{k+1}(\{b_1,\dots,b_{k+1}\})=\bigcup_{i=1}^{k+1} g^{b_i}_{k+1}(\{b_1,\dots,b_{k+1}\}).$$

Let, further, $\Phi^X_{k+1}=$

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* A family of sets $\{m_1,\dots,m_n\}$ is called a minimal covering of a set M if

$$M\subseteq\bigcup_{i=1}^n m_i$$

and

$$M\not\subseteq\bigcup_{j=1}^l m_{i_j}$$

for any proper subfamily

$$\{m_{i_1},\dots,m_{i_l}\}\subset\{m_1,\dots,m_n\}.$$

** As is easily seen, the restriction of a perfect mapping to a closed subspace is perfect.

$$=\{g^X_{k+1}(\beta),\ \beta\in B^X_{k+1}\}.$$

By the properties of g^b_{k+1} and the definition of g^X_{k+1} we have

$$g^X_{k+1}(\beta)\subseteq\bigcup_{b\in\beta} b,$$

and $g^X_{k+1}(\beta)$ contains all essential elements of β. We show that the multiplicity of Φ^X_{k+1} is $\leq\tau$. Let

$$\Phi^X_{k+1}(b)=\{g^X_{k+1}(\beta),\ \beta\ni b\}.$$

Since the family $\Phi^X_{k+1}(b)$ is elementwise* inscribed in Γ^b_{k+1}, the multiplicity of $\Phi^X_{k+1}(b)$ is $\leq\tau$. It is easy to see that the family

$$\bigcup_{b\ni x}\Phi^X_{k+1}(b)$$

exhausts all elements of Φ_{k+1}^X containing the point $x \in X$. Each of the families $\Phi_{k+1}^X(b)$ has multiplicity $\le \tau$, and there are altogether $\le \tau$ such families; consequently the multiplicity

$$\Phi_{k+1}^X \le \tau^2 = \tau.$$

Let

$$\Phi^X = \bigcup_{n=1}^{\infty} \Phi_n^X.$$

We shall not give the verification that the family

$$\{Y \setminus f(X \setminus \varphi),\ \varphi \in \Phi^X\}$$

is a certain base of the space Y and that its multiplicity is $\le \tau$. In doing this one must use the fact that the mapping f is perfect. This completes the proof of the corollary.

We now give the plan of the proof of Theorem 1.

I. Let $Y' \subseteq Y,\ X' = f^{-1}(Y')$; then $f([X']) = [f(X')]$.

II. If $Y' \subseteq Y,\ X' = f^{-1}(Y')$, and $G = \{g\}$ is a family of multiplicity $\le \tau$ of open subsets of X, then there exists a family $H = \{h\}$ of multiplicity $\le \tau$ of open subsets of X and a correspondence $\eta : G \to H$ such that

a) $\eta(g) \cap X' \subseteq g$;

b) if $y \in Y,\ f^{-1}(y) \cap [X'] \neq \emptyset$, and

$$f^{-1}(y) \cap [X' \setminus g] = \emptyset,$$

then

$$\eta(g) \supseteq f^{-1}(y).$$

III. Let B be a base of the space X of multiplicity $\le \tau$,

$$B_1 = \{f^{-1}f(b),\ b \in B\},$$

$$B_n = \{\{b_1, \dots, b_n\},\ b_1, \dots, b_n \in B_1\},$$

and $X' = f^{-1}(Y')$, where $Y' \subseteq Y$. Let, for

$$\beta = \{b_1, \dots, b_n\} \in B_n,$$

$$q(X', \beta) \cup \{f^{-1}(y),\ y \in Y,\ f^{-1}(y) \cap [X'] \neq \emptyset,\ f^{-1}(y) \cap [X' \setminus \bigcup_{b \in \beta} b] = \emptyset$$

and for every proper subfamily $\{b_{i_1}, \dots, b_{i_p}\}$ of the family $\{b_1, \dots, b_n\}$,

$$[X' \setminus (b_{i_1} \cup \dots \cup b_{i_p})] \cap f^{-1}(y) \neq \emptyset\}.$$

Then by induction on n it is proved that for every n there exists a family $\Phi_n^{X'}$ of multiplicity $\leq \tau$ of open subsets of the space X and a correspondence

$$g_n^{X'} : B_n \to \Phi_n^{X'}$$

such that

$$g_n^{X'} \supseteq q(X', \beta)$$

and

$$g_n^{X'}(\beta) \cap \left(X' \setminus \left(\bigcup_{b \in \beta} b \right) \right) = \emptyset.$$

IV. Let

$$\Phi = \bigcup_{n=1}^{\infty} \Phi_n^X.$$

Then the family

$$\{Y \setminus f(X \setminus \varphi),\ \varphi \in \Phi\}$$

is a certain base of the space Y, whose multiplicity is $\leq \tau$.

By methods analogous to those developed in the proof of Theorem 1 and its corollary, one can obtain the following results.

Theorem 2. Let $f : X \to Y$ be a perfect mapping, and let X possess a base that decomposes into τ point-finite families; then Y also possesses a base that decomposes into τ point-finite families.

Definition. A base that decomposes into τ locally finite families will be called an $NS\tau$-base.

Theorem 3. Let $f : X \to Y$ be a perfect mapping, and let X possess an $NS\tau$-base; then Y also possesses an $NS\tau$-base.

In particular, for $\tau = \aleph_0$, under the assumption that X is normal, we obtain A. H. Stone' s theorem ([3]), which states that the image of a metrizable space under a perfect mapping is metrizable.

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* The elements of the families $\Phi^X_{k+1}(b)$ and Γ^b_{k+1} are numbered by the set Δ^b_{k+1}. Elements $\varphi \in \Phi^X_{k+1}(b)$ and $\gamma \in \Gamma^b_{k+1}$ that correspond to one and the same element of Δ^b_{k+1} satisfy the relation $\varphi \subseteq \gamma$.

Note: Figure translations are in progress. See original paper for figures.