

# THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION IN THE THEORY OF ELECTROCHEMICAL MACHINING OF METAL

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Fig. 1

Figure 1: Fig. 1

## Abstract

## Full Text

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*MATHEMATICAL PHYSICS*

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# THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION IN THE THEORY OF ELECTRO-CHEMICAL MACHINING OF METAL

*(Presented by Academician S. L. Sobolev on 29 III 1967)*

**1. Formulation of the problem.** Let the tool  $D_2$  (bounded by the surface  $\Gamma_2$ ), which is the cathode, be separated from the workpiece  $D_1$  (the anode), bounded by the curve  $\Gamma_1$ , by a gap  $S$ , in which the electrolyte is located (Fig. 1). The following situation, called **stationary**, is of great practical interest: the tool  $D_2$  moves translationally parallel to the  $y$ -axis with constant velocity, and the dissolution of the workpiece  $D_1$  is such that  $\Gamma_1$  also moves parallel to the  $y$ -axis with the same velocity. As shown in <sup>(1)</sup>, in this case, in the coordinate system  $(x, y)$  “frozen” into the tool, the potential  $u(x, y)$  satisfies the equations:

$$\begin{aligned} \Delta u &= 0 \quad (\text{stationarity of the problem}); \\ u|_{\Gamma_1} &= 0, \quad u|_{\Gamma_2} = C \end{aligned} \quad (1)$$

(the workpiece and the tool are metallic);

$$\left. \frac{\partial u}{\partial n} \right|_{\Gamma_1} = \chi \cos(\vec{n}, \overrightarrow{Oy})$$

(the rate of dissolution on  $\Gamma_1$ ;  $\chi$  is a constant).

Fig. 1

The problem consists in determining, from the shape of the workpiece  $\Gamma_1$ , the shape of the tool  $\Gamma_2$ , on which the potential  $u$  assumes the prescribed value  $h$ . Generalizing somewhat, we shall consider the following problem: to find a harmonic function  $u(x, y)$  which assumes on the curve  $\Gamma_1 = \{x(\gamma), y(\gamma)\}$  the values

$$u|_{\Gamma_1} = f(\gamma), \quad \partial u / \partial n|_{\Gamma_1} = g(\gamma), \quad \gamma \in \Gamma_1. \quad (2)$$

Assume further that  $\Gamma_1$  contains an analytic piece  $\Gamma_1^a$ ; then on it  $f(\gamma)$  and  $g(\gamma)$  are real-analytic functions of  $\gamma$ . But then the solution  $u$  is determined everywhere in its domain of existence already by its values on  $\Gamma_1^a$ . Therefore we shall restrict ourselves to the case of analytic  $\Gamma_1$ .

Thus, finally, we have the problem (1)–(2) with analytic data  $\Gamma_1, f, g$ .

**Solution of problem (1)–(2).** We shall show that this problem admits an explicit solution if the functions  $f(t), g(t)$ , and  $\Gamma = \{x(t), y(t)\}$  are known to us as functions of the complex variable  $\zeta = t + is$ .

Introduce the analytic function  $w = u + iv$  of the complex variable  $z = x + iy$ . By the Cauchy–Riemann equation  $\partial u / \partial n = \partial v / \partial s$ , one can find  $v$  on  $\Gamma_1$  (up to an arbitrary constant, which is not essential). Thus the problem (1)–(2) can be reduced to the following problem:

**Problem C.** Find an analytic function  $w(z)$  having on  $\Gamma_1$  the values

$$w(t) = u(t) + iv(t), \quad \text{where } u(t) = f(t), \quad v(t) = \int_0^t g(\zeta) d\zeta.$$

Let first  $\Gamma_1$  be simply the  $x$ -axis:  $\text{Im } z = 0$ . Then the solution of problem C is

$$w(z) = u(z) + iv(z) = f(z) + i \int_0^z g(\xi) d\xi$$

and, correspondingly, of problem (1)–(2),

$$u(x, y) = \text{Re } w(z) = \frac{1}{2} [f(x + iy) + f(x - iy)] + \frac{1}{2i} \int_{x-iy}^{x+iy} g(\xi) d\xi. \quad (3)$$

(It is easy to see that the right-hand side of equality (3) is real.) Let us note that the solution (3) is an analytic continuation of d’Alembert’s formula for the wave equation, into which (2) passes if one formally sets  $y = iy_1$ .

For the case  $\Gamma = \{x(t), y(t)\}$  we shall seek the solution of problem C in parametric form, i.e., in the form  $w = w(\zeta), z = z(\zeta)$ , where  $\zeta = t + is$  is an auxiliary complex variable. On the real axis  $\text{Im } \zeta = 0$  the traces of two analytic functions are given:  $w(t) = u(t) + iv(t)$  and  $z(t) = x(t) + iy(t)$ . Thus the solution of the general problem C is reduced to the solution of two Cauchy problems with data on the real axis.

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

Of course, the solution we have obtained for problem (1)–(2), generally speaking, cannot be continued to the entire  $(x, y)$ -plane; however, this restriction is imposed not by the method, but by the essence of the problem.

Returning to the physical problem, it is easy to observe that, owing to: a) the condition of constancy of  $u(\gamma)$  on  $\Gamma_1$ , and b) the necessity of finding level lines of the function  $u(\gamma)$  in a neighborhood of  $\Gamma_1$ , the problem is technically simplified still further.

**Fig. 2**

**Discussion of the results.** In conclusion we shall describe some conclusions; although, strictly speaking, we are dealing with a plane problem, they obviously have a general meaning.

**A. Analyticity of the part.** Generally speaking, it is impossible to transmit “corners” through the electrolyte (i.e., discontinuities in some derivatives) to the tool. If a part with corners is “specified” (nonanalytic), then one must give its analytic approximation (for example, convolve the nonanalytic functions of the part with a kernel of the type

$$\exp \left[ - \left( \frac{x-y}{h} \right)^2 \right]$$

for sufficiently small  $h$ ) and consider the solution for the analytic approximation. In doing so, generally speaking, questions of correctness will arise (see, for example, (2, 3)), which we shall not consider here.

**B.** If the potential is 0, then the corresponding tool usually exists only for sufficiently small potentials  $h$  on it (in this case its shape is very close to the shape of the part); moreover, often a tool for stationary machining of a part does not exist at all: the line

the level is “knotted” for all  $h \neq 0$ . However, if the “knotting” begins at infinity, then in the finite part of the plane the line  $u = h$  gives the shape of a tool with which one can machine a part that, in the finite part of the plane, differs little from the prescribed one.

Fig. 3

**B.** We present several typical graphs (Figs. 2-4). The part is drawn with a heavy line, and the family of tools with light lines. The formula under the

Fig. 4

Figure 4: Fig. 4

drawing refers to the tool.

Fig. 4

It seems to us that, despite its triviality, the present problem is of some interest, as it provides a very adequate mathematical apparatus for an important branch of modern technical physics. The author expresses his gratitude to V. G. Shuster and R. D. Eidelman for the computations carried out at the Computing Center of the Experimental Scientific-Research Institute of Metal-Cutting Machine Tools.

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*Note: Figure translations are in progress. See original paper for figures.*

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