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Abstract

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MATHEMATICAL PHYSICS

S. G. KHARATYAN

COMMUTATIVITY OF SUPERSELECTION RULES AND A COMPLETE SET OF OBSERV- ABLES

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It was first noted by Wick, Wigner, and Wightman in ⁽¹⁾ that not all rays in Hilbert space correspond to physically realizable states. Indeed, for example, a superposition of states with integer and half-integer spin in a theory invariant with respect to rotations is physically unrealizable. It follows from this that not all Hermitian operators represent observables, and the algebra of observables may be reducible. Any assertion that singles out certain rays as physically unrealizable is called a superselection rule.

As is clear, any physically realizable state must be an eigenstate of the operators Q , B , and $(-1)^F$, where Q is the electric charge, B is the baryon number, and F is even for states with integer spin and odd for states with half-integer spin. The operators Q , B , and $(-1)^F$ specify superselection rules for charge, baryon number, and univalence. If there are no other superselection rules, then, by virtue of the commutativity of the operators Q , B , $(-1)^F$, the entire Hilbert space is a direct sum of subspaces in which all rays correspond to realizable states and the superposition principle is valid. In the case of arbitrary superselection rules it has been postulated that the entire Hilbert space is a direct sum, generally speaking a direct integral, of "coherent subspaces," i.e., subspaces in which all rays correspond to realizable states, all observables are defined as operators, and the algebra of observables is irreducible. In works ^(2, 3) the concept of commutativity of superselection rules was introduced. If all operators belonging to the commutant of the observables commute with one another, then the superselection rules are said to be commutative. From the fact that then all Hermitian operators belonging to the commutant of the observables are functions of a certain Hermitian operator and can be simultaneously diagonalized, it follows that the entire Hilbert space is a direct integral of subspaces in which all observables are defined as operators and the algebra of observables is irreducible. If, as has always been done, one assumes that all superselection rules are specified by operators, then these subspaces, obviously, will be "coherent."

In work ⁽⁵⁾ it was noted that results equivalent to the results of works ^(2, 3)

were obtained by Jauch in work (4). As shown in (5), the main assumption of work (4) is equivalent to the assumption that the superselection rules are commutative.

It will be shown that the Hilbert space \mathcal{H} describing a physical system is a direct sum of subspaces \mathcal{H}_I and \mathcal{H}_{II} such that in \mathcal{H}_I the commutativity of the superselection rules holds, while in \mathcal{H}_{II} there is not a single vector that corresponds to a physically realizable state. \mathcal{H}_{II} may in general be discarded.

Define \mathcal{H}_I as the closure of the linear manifold spanned by all vectors that correspond to physically realizable states. Then $\varphi \in \mathcal{H}_I$, $\varphi = \sum_i a_i \psi_i$, where all ψ_i are realizable. It should be noted,

that, generally speaking, different decompositions with respect to different ψ_i may correspond to all φ , and that in these decompositions the ψ_i may also be nonorthogonal. The projector P_ψ , if ψ corresponds to a realizable state, represents an observable. Every operator commuting with all observables must necessarily commute with all P_ψ , where ψ is realizable. It will be proved that two operators commuting with all P_ψ , where ψ is realizable, commute with each other. Obviously, it is sufficient to prove this assertion for projectors. Let $P_{\mathfrak{M}_1}$ and $P_{\mathfrak{M}_2}$ be two such projectors. Then every realizable ψ either belongs to \mathfrak{M}_1 , or is orthogonal to it, since if $P_{\mathfrak{M}}$ commutes with $P_{\mathfrak{S}}$, then $P_{\mathfrak{M}}P_{\mathfrak{S}} = P_{\mathfrak{M} \cap \mathfrak{S}}$.

If $\varphi \in \mathcal{H}_1$, then

$$P_{\mathfrak{M}_1}P_{\mathfrak{M}_2}\varphi = P_{\mathfrak{M}_1}P_{\mathfrak{M}_2}\left(\sum_k a_k\psi_k\right),$$

where all ψ_k are realizable. Therefore we have $P_{\mathfrak{M}_i}\psi_k = \psi_k$, $\psi_k \in \mathfrak{M}_i$, or $P_{\mathfrak{M}_i}\psi_k = 0$. Then

$$P_{\mathfrak{M}_1}P_{\mathfrak{M}_2}\varphi = \sum_k' a_k\psi_k,$$

where the sum contains only $\psi_k \in \mathfrak{M}_1 \cap \mathfrak{M}_2$.

Analogously it is proved that

$$P_{\mathfrak{M}_2}P_{\mathfrak{M}_1}\varphi = \sum_k' a_k\psi_k,$$

where the sum contains only ψ_k belonging to $\mathfrak{M}_1 \cap \mathfrak{M}_2$. Hence $P_{\mathfrak{M}_1}P_{\mathfrak{M}_2} = P_{\mathfrak{M}_2}P_{\mathfrak{M}_1}$, and the commutativity of the superselection rules is proved.

Let us emphasize that in the proof only two assumptions were used: 1) the realizable states of the system correspond to rays in some Hilbert space; 2) if

ψ corresponds to a realizable state, then P_ψ represents an observable, and not even separability of the Hilbert space was assumed.

If we assume that all superselection rules are given by operators with discrete spectrum, then, as was already noted above, the entire Hilbert space is represented as a direct sum of coherent subspaces.

In ⁽³⁾ it was noted that the superselection rules will be commutative if there exists a complete set of observables. We shall prove that if the whole space decomposes into a direct sum of coherent subspaces, then a complete set exists.

Suppose that the coherent spaces are separable. Then in each of them one can construct a countable, complete orthonormal system. Let ψ_{ik} be the i -th vector of the system spanning the k -th coherent subspace.

Consider the projectors $P_{\psi_{ik}}$. They all commute, since the ψ_{ik} are pairwise orthogonal, and they are all observable, since all ψ_{ik} are realizable. Consider the operator

$$P = \sum_{ik} \lambda_{ik} P_{\psi_{ik}}, \quad 0 < \lambda_{ik} < 1, \quad \lambda_{ik} \neq \lambda_{mn},$$

$$(i, k) \neq (m, n).$$

This operator will play the role of a complete set. It will be shown that every Hermitian operator which commutes with P is a function of P , which, obviously, is sufficient to prove for an arbitrary projector.

Indeed, let some $P_{\mathfrak{M}}$ commute with P . Then, by the construction of P , it will commute with all $P_{\psi_{ik}}$. It follows from this that each vector ψ_{ik} either lies in \mathfrak{M} , or is orthogonal to it. Then, since the ψ_{ik} form a complete system, \mathfrak{M} will represent the closed linear manifold spanned by those ψ_{ik} which lie in it.

$$P_{\mathfrak{M}} = \sum_{ik} P_{\psi_{ik}},$$

where the sum contains those i, k such that $\psi_{ik} \in \mathfrak{M}$. Hence it follows that $P_{\mathfrak{M}} = f(P)$, where f is the operator function corresponding to the function

$$f(\lambda) = \begin{cases} 1, & \lambda = \lambda_{ik}, \psi_{ik} \in \mathfrak{M}, \\ 0 & \text{for all other values of } \lambda. \end{cases}$$

Consequently, the existence of a complete set has been proved.

The principal result of the present work should be regarded as the proof of the commutativity of superselection rules. It should be noted that this proof became possible thanks to the decomposition of \mathcal{H} into \mathcal{H}_I and \mathcal{H}_{II} , since all

operators lying in \mathcal{H}_{Π} and equal to 0 in \mathcal{H}_{Π} commute with all observables, but do not commute with one another if \mathcal{H}_{Π} is more than one-dimensional.

In the proof we used the postulate that, if ψ corresponds to a realizable state, then P_{ψ} represents an observable. In works ⁽¹⁻⁵⁾, in the study of superselection rules, this postulate was not invoked.

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Moscow State University
named after M. V. Lomonosov

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