ON THE UNCONDITIONAL EXTREMUM OF FUNCTIONALS AND ON THE CONVERGENCE OF MINIMIZING SEQUENCES

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Abstract

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ON THE UNCONDITIONAL EXTREMUM OF FUNCTIONALS AND ON THE CONVERGENCE OF MINIMIZING SEQUENCES

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The following is due to A. N. Tikhonov (1).

Definition. The problem of minimizing a real functional defined on a certain subset of a vector space is posed correctly if it is solvable, has a unique solution, and every minimizing sequence converges to it in the sense of the strong topology of the space.

Here we consider the problem of the unconditional minimum of functionals defined in real Banach spaces, and give sufficient conditions for its well-posedness in the sense of A. N. Tikhonov.

1. Let a real Banach space E be contained in some real Hilbert space H, where H is contained densely in the conjugate space E^*. We shall also assume that condition (α) of (2) is satisfied. In the case of a Banach space this means that (3)

$$\|x\|_H \le a\|x\|_E \quad \text{for } x \in E, \qquad \|y\|_{E^*} \le b\|x\|_H \quad \text{for } y \in H$$

$(a > 0,\ b > 0)$, and the value of the linear functional $y \in E^*$ on the vector $x \in E$, i.e. (y, x), coincides with the scalar product in H for $y \in H$. Such Banach spaces are given, for example, in (2,3).

Let, further, F be a nonlinear continuous operator from E into E^*, and let B be a bounded operator from E^* into E, whose restriction B_H to H is a self-adjoint operator in H.

Consider the nonlinear functional

$$f(x) = \|x - BF(x)\|, \qquad x \in E$$

and pose the problem of its absolute minimum.

Theorem 1. *If $(-1)F$ is a monotone operator, i.e.*

$$(F(x) - F(y),\, x - y) \le 0, \qquad x, y \in E,$$

and B_H is a positive operator in H, then the problem of minimizing $f(x)$ on H is posed correctly.

Theorem 2. *If B_H is a quasi-negative (^{4}) operator in H, and F is a monotone operator satisfying the condition*

$$(F(x)-F(y),\, x-y) \geq 2\lambda \|x-y\|^2, \qquad x, y \in E,$$

where λ is the largest characteristic number of the operator B_H, then the problem of minimizing $f(x)$ on E is posed correctly.

The proof of Theorems 1 and 2 uses considerations from ($^{4-6}$), in which potential and nonpotential monotone operators were first studied.

We note that the requirement of continuity of the operator F can be weakened. It is enough to require its hemicontinuity (^{7}).

2. Theorems 1 and 2 lead to new propositions on the existence and uniqueness of the solution of the equation $x = BF(x)$ and give an approximate method for finding its solution. We present one such proposition.

Let the following conditions be fulfilled:

I. $E = L_{p,n}(D)$, i.e. E is the direct sum of n Lebesgue spaces $L_p(D)$, where $p > 2$ and D is a set of finite measure in s-dimensional Euclidean space. Then $H = L_{2,n}$ and $E^* = L_{q,n}$ ($p^{-1} + q^{-1} = 1$).

II. B_{ij} is a linear integral bounded operator from L^q into L^p with kernel $K_{ij}(x,y)$ ($x, y \in D$; $i, j = 1, 2, \dots, n$), $K(x,y) = (K_{ij}(x,y))$ is the square matrix of kernels $K_{ij}(x,y)$, and B is the integral operator with kernel $K(x,y)$, which, according to the condition, acts from $L_{q,n}$ into $L_{p,n}$.

III. The Nemytskii operator $h = (h_1, h_2, \dots, h_n)$, $h_i u = g_i(u_1(x), \dots, u_n(x), x)$ (see (4)) acts from $L_{p,n}$ into $L_{q,n}$, and its potentiality is not assumed.

IV. B_H ($H = L_{2,n}$) is a quasipositive operator.

V. For almost all $x \in D$ and arbitrary real u_i and v_i ($i = 1, 2, \dots, n$),

$$\sum_{i=1}^{n} [g_i(u_1+v_1, \dots, u_n+v_n, x) - g_i(u_1, \dots, u_n, x)] v_i \geq 2\lambda \sum_{i=1}^{n} v_i^2,$$

i.e. the operator h satisfies the same condition as F in Theorem 2.

Theorem 3. *If conditions I–V are fulfilled, then the system of nonlinear integral equations*

$$u_i(x) = \sum_{j=1}^{n} B_{ij} h_j u \qquad (i = 1, 2, \dots, n) \tag{1}$$

has a unique solution belonging to the space $L_{p,n}$.

An analogous proposition holds for the system (1) in a space that is the direct sum of the Orlicz spaces considered in (^{3}).

3. Let E be a Banach space with a differentiable norm. Examples of such spaces are the sequence spaces l^p and the Lebesgue spaces L^p $(p > 1)$. Consider the duality mapping

$$U(x) = \|x\| \operatorname{grad} \|x\|, \qquad U(0) = 0,$$

from E into E^*. As is known (^{6},^{9}), it has the properties

$$(U(x), x) = \|x\|^2, \qquad \|U(x)\| = \|x\|,$$

where (z, x) is the value of the linear functional $z \in E^*$ on the vector $x \in E$.

Let F be a nonlinear continuous operator from E into E, and let v be an arbitrary fixed vector in E. We pose the problem of the absolute minimum of the functional

$$f_v(x) = \|F(x) - v\|, \qquad x \in E,$$

i.e. of finding a vector x_v for which $f_v(x_v) = 0$.

Theorem 4. *Suppose that the condition*

$$(U(x-y), F(x) - F(y)) \geq m(r)\|x-y\|^2, \qquad x, y \in D_r = \{x : \|x\| \leq r\},$$

is fulfilled, where $m(t)$ is a positive decreasing function defined for $t \geq 0$ and

$$\lim_{t \to +\infty} tm(t) = +\infty, \qquad \underline{\lim}\, tm(t) > 0.$$

Then the problem of the absolute minimum of the functional $f_v(x)$ is well posed.

4. Let E be a reflexive Banach space and let $f(x)$ be a real-valued functional defined on E and differentiable in the sense of Gâteaux. Put $F(x) = \operatorname{grad} f(x)$. First we shall give sufficient conditions for convergence of a minimizing sequence to the minimum point x_0 of the func-

of the functional $f(x)$, and then formulate other propositions. Since by means of a shift one may assume $x_0 = 0$, in the following proposition we shall assume $x_0 = 0$.

Theorem 5. Suppose that $f(0) = \min f(x)$, $(F(x), x) \geq \|x\|\gamma(\|x\|)$, where $(F(tx), \dot{x})$ is integrable with respect to t on $[0, 1]$, $\gamma(t)$ is integrable on $[0, R]$ for every $R > 0$, and the function

$$c(R) = \int_0^R \gamma(t)\,dt \tag{2}$$

is increasing.

Then every minimizing sequence converges strongly to zero, and the problem of minimizing $f(x)$ is well posed.

Theorem 6. Suppose that the operator $F(x) = \operatorname{grad} f(x)$ satisfies the conditions: for all $h, y \in E$ the function $(F(y+th), h)$ is integrable with respect to t on $[0, 1]$, and

$$(F(y+h) - F(y), h) \geq \|h\|\gamma(\|h\|),$$

where the function $c(R)$, defined by equality (2), is increasing, and there exists an R such that $c(R) > R\|F(0)\|$.

Then $f(x)$ has a unique point of minimum, and the problem of minimizing $f(x)$ is well posed.

Theorem 7. Suppose that a twice Gâteaux differentiable functional $f(x)$ satisfies the conditions: $D^2 f(tx; x, x)$ is integrable with respect to t on $[0, 1]$ for every $x \in E$, and

$$D^2 f(x; h, h) \leq \gamma(\|x\|)\|h\|^2,$$

where $\gamma(z)$ is a positive decreasing function for $z \geq 0$ such that, for some $R > 0$,

$$\frac{1}{R}\alpha(R) \equiv \frac{1}{R}\int_0^R z\gamma(z)\,dz > \|F(0)\|, \qquad F(x) = \operatorname{grad} f(x). \tag{3}$$

Then the problem of minimizing $f(x)$ is well posed.

We note that, for inequality (3) to hold, it is sufficient that

$$\varlimsup_{R\to+\infty} \frac{1}{R}\alpha(R) = +\infty.$$

5. We give further sufficient conditions for the minimum of a real-valued functional $f(x) \neq +\infty$ defined on a reflexive Banach space (cf. (4)).

Theorem 8. Suppose that a weakly lower semicontinuous functional $f(x)$, Gâteaux differentiable, $\operatorname{grad} f(x) = F(x)$, is continuous along every segment tx $(0 \leq t \leq 1)$, and $(F(x), x) \geq \|x\|\gamma(\|x\|)$, where the function $\gamma(t)$ is integrable on $[0, R]$ for every $R > 0$.

Then, if

$$\overline{\lim_{R\to+\infty}} \int_0^R \gamma(z)\,dz > 0,$$

there exists a point of minimum of $f(x)$.

Theorem 9. Suppose that a weakly lower semicontinuous functional $f(x)$ satisfies the condition

$$\lim_{R\to+\infty} f(x) = +\infty \qquad (R = \|x\|). \tag{4}$$

Then there exists a point of minimum of $f(x)$.

We note that if $f(x)$ is differentiable with respect to I and $F(x) = \operatorname{grad} f(x)$, then for condition (4) to hold it suffices that

$$\lim_{R\to+\infty} \int_0^1 (F(tx), x)\,dt = +\infty \qquad (R = \|x\|).$$

In conclusion, let us note that, for the construction of minimizing sequences, one may use the method of steepest descent ([9]), Ritz' s method ([10]), and various iterative methods (see, for example, ([11])).

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Note: Figure translations are in progress. See original paper for figures.