AN OPERATIONAL METHOD FOR SOLVING MIXED PROBLEMS FOR LINEAR DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE WITH DISCONTINUOUS COEFFICIENTS

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Abstract

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AN OPERATIONAL METHOD FOR SOLVING MIXED PROBLEMS FOR LINEAR DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE WITH DISCONTINUOUS COEFFICIENTS

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The mixed problem (1)–(3) was investigated in the general case of complex-valued functions $c_2^{(i)}(x)$, $-\pi/2 < \arg c_2^{(i)}(x) < \pi/2$ $(i = 1, \dots, n)$, by the contour-integral method in the monograph of M. L. Rasulov ([1]). We consider an operational scheme of solution and give a justification of the operational method for solving problem (1)–(3) on the half-line $t > 0$. The realization of the obtained operational formulas of solution (8) is connected with the inversion of Laplace integrals. Numerical methods for inverting these integrals have been sufficiently developed ([5], Ch. 5).

1. Consider the problem of finding solutions of the equations

$$\frac{\partial u^{(i)}(x,t)}{\partial t} = \sum_{l=0}^{2} c_l^{(i)}(x) \frac{\partial^l u^{(i)}(x,t)}{\partial x^l} + f^{(i)}(x,t), \tag{1}$$

$$x \in (a_i, b_i), \quad t \in (0, T], \quad (i = 1, \dots, n)$$

under the boundary conditions

$$\sum_{i=1}^{n} \sum_{l=0}^{1} \left\{ \alpha_{sl}^{(i)} \left(\frac{\partial}{\partial t}\right) \left.\frac{\partial^l u^{(i)}(x,t)}{\partial x^l}\right|_{x=a_i} + \beta_{sl}^{(i)} \left(\frac{\partial}{\partial t}\right) \left.\frac{\partial^l u^{(i)}(x,t)}{\partial x^l}\right|_{x=b_i} \right\} = \gamma_s,$$

$$t \in (0, T], \quad (s = 1, 2, \dots, 2n) \tag{2}$$

and the initial conditions

$$u^{(i)}(x, +0) = 0, \quad x \in (a_i, b_i) \quad (i = 1, \dots, n), \tag{3}$$

where

$$\alpha_{sl}^{(i)}\left(\frac{\partial}{\partial t}\right)=\alpha_{sl0}^{(i)}+\alpha_{sl1}^{(i)}\frac{\partial}{\partial t},\qquad \beta_{sl}^{(i)}\left(\frac{\partial}{\partial t}\right)=\beta_{sl0}^{(i)}+\beta_{sl1}^{(i)}\frac{\partial}{\partial t},$$

$\alpha_{slk}^{(i)}$, $\beta_{slk}^{(i)}$ $(l,k=0,1)$, γ_s $(s=1,2,\dots,2n)$ are, generally speaking, complex constants; (a_i,b_i) are mutually nonintersecting intervals having common boundary points and contained in the basic interval (a_1,b_n) $(n\geq 1)$.

Problem (1)–(3) is reduced to two problems: **problem A**—the determination of solutions $v^{(i)}(x,t)$ of the homogeneous equations corresponding to equation (1), under the boundary conditions (2) and the initial conditions (3), and **problem B**—the determination of solutions $w^{(i)}(x,t)$ of equations (1) satisfying the homogeneous boundary conditions obtained from (2) when $\gamma_s=0$ $(s=1,2,\dots,2n)$, and the initial conditions (3).

By a solution of problems A and B we shall mean functions $v^{(i)}(x,t)$ and $w^{(i)}(x,t)$ $(i=1,\dots,n)$ that are continuous for $x\in(a_i,b_i)$, $t\in[0,T]$, and for $x\in[a_i,b_i]$, $t\in(0,T]$ $(0<T<\infty)$, all derivatives entering into (1) are continuous with respect to $x\in(a_i,b_i)$ and $t\in(0,T]$, while the derivatives entering into (2) are continuous with respect to $x\in[a_i,b_i]$ and $t\in(0,T]$.

We shall assume the following conditions to be fulfilled:

a) the functions $c_l^{(i)}(x)$ $(l=0,1,2)$ are $6-l$ times continuously differentiable on $[a_i,b_i]$, with $c_0^{(i)}(x)$, $c_1^{(i)}(x)$ complex-valued, and $c_2^{(i)}(x)$ real-valued functions, and $c_2^{(i)}(x)>0$ for all $x\in[a_i,b_i]$;

b) for any $t\in[0,T]$ the functions $\partial^k f^{(i)}(x,t)/\partial t^k$ $(k=0,1,2)$ and $\partial f^{(i)}(x,t)/\partial x$ are continuously differentiable with respect to x on $[a_i,b_i]$; $f^{(i)}(x,t)$ are complex-valued functions.

Taking for $z=\sqrt{\lambda}$ that branch for which $z\in\widetilde{\Pi}_1\{z:0\leq\arg z\leq\pi/2\}$ when $\lambda\in\Pi_1\{\lambda:0\leq\arg\lambda\leq\pi\}$ and $z\in\widetilde{\Pi}_2\{z:-\pi/2\leq\arg z\leq 0\}$ when $\lambda\in\widetilde{\Pi}_2\{\lambda:-\pi\leq\arg\lambda\leq 0\}$, put

$$A_{sk}^{(i)}(z)=\sum_{l=0}^{1}\alpha_{sl}^{(i)}(z^2)\big(z\varphi_k^{(i)}(a_i)\big)^l,\quad B_{sk}^{(i)}(z)=$$

$$=\sum_{l=0}^{1}\beta_{sl}^{(i)}(z^2)\big(z\varphi_k^{(i)}(b_i)\big)^l,\quad \varphi_k^{(i)}(x)=(-1)^k\big(c_2^{(i)}(x)\big)^{-1/2}\quad(k=1,2;\ s=1,2,\dots,2n).$$

Denoting by $A_k^{(i)}(z)$, $B_k^{(i)}(z)$ $(k=1,2)$ the columns with elements $A_{1k}^{(i)}(z),\dots,A_{2n,k}^{(i)}(z)$ and $B_{1k}^{(i)}(z),\dots,B_{2n,k}^{(i)}(z)$, respectively, we formulate conditions A and $\mathrm{B}_{(k)}$ $(k=0,1,2)$.

A. The determinants

$$\left|A_1^{(1)}(z)\ B_2^{(1)}(z)\ A_1^{(2)}(z)\ B_2^{(2)}(z)\ \dots\ A_1^{(n)}(z)\ B_2^{(n)}(z)\right|,$$

$$\left|B_1^{(1)}(z)\ A_2^{(1)}(z)\ B_1^{(2)}(z)\ A_2^{(2)}(z)\ \dots\ B_1^{(n)}(z)\ A_2^{(n)}(z)\right|$$

are polynomials in z of the same degree $d \geq 0$, different from the identically zero polynomial. All determinants of order $2n$ formed from other combinations of the columns $A_k^{(i)}(z), B_k^{(i)}(z)$ $(k = 1, 2;\ i = 1, \dots, n)$ are polynomials in z of degree not exceeding d.

$\mathbf{B}_{(k)}$**.** Suppose that in condition A $d \geq k$ and all determinants of order $2n - 1$ formed from the elements of the matrix

$$\left(A_1^{(1)}(z)\ A_2^{(1)}(z)\ \dots\ A_1^{(n)}(z)\ A_2^{(n)}(z)\ B_1^{(1)}(z)\ B_2^{(1)}(z)\ \dots\ B_1^{(n)}(z)\ B_2^{(n)}(z)\right)$$

are polynomials in z of degree not exceeding $d - k$.

To the original problem (1)–(3) we assign the boundary-value problem with a complex parameter (the spectral problem)

$$\sum_{l=0}^{2} c_l^{(i)}(x) \frac{d^l y^{(i)}}{dx^l} - \lambda y^{(i)} = 0, \quad a_i < x < b_i \quad (i = 1, \dots, n); \tag{4}$$

$$\sum_{i=1}^{n} \sum_{l=0}^{1} \left\{ \alpha_{sl}^{(i)}(\lambda) \frac{d^l y^{(i)}}{dx^l}\bigg|_{x=a_i} + \beta_{sl}^{(i)}(\lambda) \frac{d^l y^{(i)}}{dx^l}\bigg|_{x=b_i} \right\} = \gamma_s \quad (s = 1, 2, \dots, 2n). \tag{5}$$

Let $y_k^{(i)}(x, \lambda)$ $(k = 1, 2)$ be a fundamental system of particular solutions of the i-th equation (4), consisting of functions which, together with their first derivatives with respect to x, are entire functions of the parameter λ (see Poincaré' s theorem ([4]), p. 28). Denote

$$u_{sk}^{(i)}(\lambda) = \sum_{l=0}^{1} \left\{ \alpha_{sl}^{(i)}(\lambda) \frac{d^l y_k^{(i)}}{dx^l}\bigg|_{x=a_i} + \beta_{sl}^{(i)}(\lambda) \frac{d^l y_k^{(i)}}{dx^l}\bigg|_{x=b_i} \right\}$$

$$(k = 1, 2;\ i = 1, \dots, n;\ s = 1, 2, \dots, 2n).$$

Now making use of the auxiliary notation from ([1]), we introduce for consideration, by formulas (6.2.5), (6.2.12)–(6.2.16) of ([1]), the functions $w^{(i)}(x, \lambda)$,

$\Delta^{(i)}(x,\lambda)$, $\Delta^{(i,j)}(x,\xi,\lambda)$, $g^{(i,j)}(x,\xi,\lambda)$, $\Delta(\lambda)$, $G^{(i,j)}(x,\xi,\lambda)$. Following Birkhoff ([3]), we shall denote a sum of the form $f(x)+E(x,z)/z$, where $E(x,z)$ is continuous in x and bounded for $|z|\geq R$ (R is a sufficiently large positive number), by the symbol $[f(x)]$. Then, applying Tamarkin' s theorem on the asymptotic representation of a certain fundamental system of particular solutions to each i-th equation of (4) (see ([2]), theorem 3), for the determinant $\widetilde{\Delta}(z)=\Delta(z^2)$ we find, for all sufficiently large in modulus $z\in\widetilde{\Pi}$, $\widetilde{\Pi}=\widetilde{\Pi}_1+\widetilde{\Pi}_2$, the asymptotic formula

$$\widetilde{\Delta}(z)=z^{d-n}H(z)\prod_{i=1}^{n}\psi^{(i)}(a_i z^2),$$

$$H(z)=\sum_{\mu=1}^{\sigma}[M_\mu]\exp(m_\mu z),\qquad m_1<\dots<m_\sigma,$$

$$-m_1=m_\sigma=\sum_{i=1}^{n}\int_{a_i}^{b_i}\left(c_2^{(i)}(x)\right)^{-1/2}dx,\qquad M_1\neq 0,\qquad M_\sigma\neq 0.$$

By Lemma 1 of ([1]), all zeros of the function $H(z)$ are located in the z-plane in a certain strip D_h, bounded by the straight lines $\mathrm{Re}(z)=\pm h$ ($h>0$).

With the aid of the concrete asymptotics of the solution of the spectral problem (4)–(5) and of the existence and well-posedness theorems for the solutions of problems A and B (Theorems 24, 25, 28 of ([1])), we prove the following theorems.

Theorem 1. *Under conditions a), b) and A, problem B has a unique solution $w^{(i)}(x,t)$, depending continuously on the right-hand sides of equations (1) and representable for all $x\in[a_i,b_i]$, $t\in(0,T]$ and $x\in(a_i,b_i)$, $t\in(0,T]$ by the formula*

$$w^{(i)}(x,t)=-\frac{1}{2\pi\sqrt{-1}}\lim_{\omega\to\infty}\int_{\eta-\omega\sqrt{-1}}^{\eta+\omega\sqrt{-1}}e^{\lambda t}\,d\lambda\sum_{j=1}^{n}\int_{a_j}^{b_j}G^{(i,j)}(x,\xi,\lambda)\times$$

$$\times\int_0^t e^{-\lambda\tau}f^{(j)}(\xi,\tau)\left(c_2^{(j)}(\xi)\right)^{-1}d\tau\,d\xi,\qquad \eta>h^2\quad(i=1,\dots,n).\tag{6}$$

All derivatives of $w^{(i)}(x,t)$ entering into problem B are obtained by differentiating under the sign of the contour integral in (6).

Let us call $A_{(0)}$, $A_{(1)}$, $A_{(2)}$ the problems A with boundary conditions (2), respectively: 1) containing no derivatives with respect to t; 2) containing the derivative $\partial u^{(i)}/\partial t$ for at least one value of i and s; 3) containing the derivative $\partial^2 u^{(i)}/\partial t\,\partial x$ for at least one value of i and s.

Theorem 2. *Problem $A_{(k)}$ has, under conditions a), A, $B_{(k)}$, a unique solution, depending continuously on the right-hand sides of the boundary conditions (2) and representable for all $x \in [a_i, b_i]$, $t > 0$ and $x \in (a_i, b_i)$, $t \geq 0$ by the formula*

$$v^{(i)}(x,t) = \frac{1}{2\pi\sqrt{-1}} \lim_{\omega\to\infty} \int_{\eta-\sqrt{-1}\omega}^{\eta+\sqrt{-1}\omega} e^{\lambda t} \frac{\Delta^{(i)}(x,\lambda)}{\lambda\Delta(\lambda)}\, d\lambda, \qquad \eta > h^2 \quad (i = 1, \dots, n) \tag{7}$$

All derivatives of the solution $v^{(i)}(x,t)$ entering into problem $A_{(k)}$ are obtained by differentiating under the sign of the contour integral in (7).

2. Usually, when solving problem (1)–(3) on the half-line $t > 0$ by the operational method (with the aid of the one-sided Laplace transform with respect to the variable t), one assumes that the right-hand sides of equations (1) are Laplace-transformable with respect to t and makes the following assumptions: 1) the solution of the problem and all derivatives of the solution entering into the formulation of the problem are Laplace-transformable; 2) the operations of differentiation $\partial/\partial x$, $\partial^2/\partial x^2$ commute with L_t (L_t is the Laplace integral); 3) the operations of passage to the limit as $x \to a_i \pm 0$, $x \to b_i \pm 0$ commute with L_t. Naturally, by this route only a formal solution can be obtained. Thus, for example, problem (1)–(3) is solved in [6] with coefficients $c_l^{(i)}(x)$ ($l = 0, 1, 2;\ n = 2$), independent of x, and with boundary conditions containing no derivatives with respect to t.

Let $S\{a, b\}$ be the set of all functions $\Phi(x,t)$, defined for $x \in [a,b]$ (or $x \in (a,b)$), $0 \leq t < \infty$, integrable with respect to t on every finite interval $[0,T]$ for any $x \in [a,b]$ (respectively, $x \in (a,b)$), and for which the Laplace integral L_t converges at some point $\lambda_0 = \lambda_0(\Phi)$ uniformly with respect to $x \in [a,b]$ (respectively, $x \in$

$\in (a,b)$). To each fixed function $\Phi(x,t) \in S\{a,b\}$ there corresponds the set $E\{\Phi, a, b\}$ of all real values λ for which L_t converges uniformly with respect to x. The lower bound of the set $E\{\Phi, a, b\}$ will be called the **abscissa of convergence** of the integral $L_t\{\Phi(x,t)\}$.

Theorem 3. Under conditions a), A, $B_{(k)}$, the solution $v^{(i)}(x,t)$ of problem $A_{(k)}$, together with all derivatives entering this problem, belongs to the set $S\{a_i, b_i\}$. The abscissa of convergence of their Laplace integrals does not exceed h^2.

Theorem 4. Let the right-hand sides $f^{(i)}(x,t)$ of equations (1) be continuous for $x \in [a_i, b_i]$, $t > 0$, and let, on the interval $[a_i, b_i]$, all the functions

$$\frac{\partial^{s+m}}{\partial t^s \partial x^m} f^{(i)}(x,t)$$

$(m = 0, 1, 2;\ s = 0, 1)$ belong to the set $S\{a_i, b_i\}$ with abscissa of convergence $\mu^{(i)} \geqslant 0$ of their Laplace integrals.

Then, under conditions a), A, the solution $w^{(i)}(x,t)$ of problem B, together with all derivatives entering this problem, belongs to the set $S\{a_i, b_i\}$. The abscissa of convergence of their Laplace integrals does not exceed $\max(\mu^{(i)}, h^2)$.

It is clear that, under the conditions of Theorems 3 and 4, assumptions 1), 2), 3) are certainly satisfied for problem (1)–(3), which makes it possible to solve this problem directly by the operational method.

As a result, for the solution of the problem,

$$u^{(i)}(x,t) = \frac{1}{2\pi\sqrt{-1}} \lim_{\omega\to\infty} \int_{\eta-\omega\sqrt{-1}}^{\eta+\omega\sqrt{-1}} e^{\lambda t} \times$$

$$\times \left(\frac{\Delta^{(i)}(x,\lambda)}{\lambda\Delta(\lambda)} - \sum_{j=1}^{n} \int_{a_j}^{b_j} G^{(i,j)}(x,\xi,\lambda)\, \bar{f}^{(j)}(\xi,\lambda)\, \left(c_2^{(j)}(\xi)\right)^{-1} d\xi \right) d\lambda, \qquad (8)$$

$$\eta > \max(\mu^{(i)}, h^2) \qquad (i = 1, \dots, n),$$

which is valid for all $x \in [a_i, b_i]$, $t > 0$, and $x \in (a_i, b_i)$, $t \geq 0$.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.