ABSOLUTE CONTINUITY OF FUNCTIONS OF THE CLASS $\widetilde W_p^1$ ON LEVEL SETS OF A FUNCTION OF THE CLASS $\widetilde W_q^1$ AND SOME BOUNDARY PROPERTIES OF MAPPINGS WITH GENERALIZED DERIVATIVES IN A PLANE DOMAIN

Abstract

Full Text

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MATHEMATICS

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ABSOLUTE CONTINUITY OF FUNCTIONS OF THE CLASS $\widetilde{W}_p^1$ ON LEVEL SETS OF A FUNCTION OF THE CLASS $\widetilde{W}_q^1$ AND SOME BOUNDARY PROPERTIES OF MAPPINGS WITH GENERALIZED DERIVATIVES IN A PLANE DOMAIN

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Let G be an arbitrary domain of the plane R^2. Denote by $W_p^1(G)$ (respectively $W_\infty^1(G)$) the class of functions $r(z)$ continuous in G, $z = (x, y)$, having there first-order generalized derivatives in the sense of S. L. Sobolev (see ([1,2])), summable to the power $p \geqslant 1$ (respectively essentially bounded) on every compact set $K \subset G$. The totality of functions of the class $W_p^1(G)$ differentiable almost everywhere in G will be denoted by $\widetilde{W}_p^1(G)$.

Let $T(z)$ be a mapping of G into R^2, $T = u + iv$. We shall say that $T \in \widetilde{W}_p^1(G)$ if $u, v \in \widetilde{W}_p^1(G)$.

As is known, a function of the class $W_1^1(G)$ is absolutely continuous inside almost all x-sections and y-sections of the domain G. In the present work a more general fact is established, with the aid of which some boundary properties of mappings $T \in \widetilde{W}_p^1(G)$ are proved, in particular, of $\widetilde{BL}$-homeomorphisms, to which the monograph of G. D. Suvorov ([3]) is devoted.

If the domain G is finitely connected, then the definitions and theorems given below in Sec. 3 can be modified for sets of simple ends of G, for example as was done in the paper ([4]).

1. If a function $r \in \widetilde{W}_1^1(G)$, then the orthogonal vectors $\nabla r(z)$ and $s(z) = (-\partial r/\partial y,\ \partial r/\partial x)$ are defined almost everywhere in G.

For a function $f \in \widetilde{W}_1^1(G)$ and a vector field s we put in G almost everywhere

$$D_s f \overset{\text{def}}{=} \begin{cases} \left(-\dfrac{\partial f}{\partial x}\dfrac{\partial r}{\partial y} + \dfrac{\partial f}{\partial y}\dfrac{\partial r}{\partial x}\right) \cdot |\nabla r|^{-1}, & \text{if } \nabla r \neq 0, \\ 0, & \text{if } \nabla r = 0. \end{cases}$$

If a mapping $T \in \widetilde{W}_1^1(G)$, then, by definition, $D_{sT} = D_{su} + iD_{sv}$.

A plane curve is a continuous mapping $z : I \to R^2$, where I is a closed interval of the numerical axis. If $\Gamma = z(I)$, and the mapping $T : \Gamma \to R^2$ is continuous, then by the symbol $l[T(\Gamma)]$ we denote the length of the curve $T \circ z$.

Further, if r is a function defined in G, and $A \subset G$, then

$${}_tA \overset{\text{def}}{=} \{z \in A \mid r(z) = t\}.$$

Let $t_1 = \inf r(z)$ and $t_2 = \sup r(z)$, $z \in G$.

Throughout the sequel H denotes linear Hausdorff measure; $\overline{M}$ is the closure of the set $M \subset R^2$; $\operatorname{Fr} M$ is the boundary of M.

From the results of A. S. Kronrod ([5]), taking into account the paper ([6]), it follows that for $r \in \widetilde{W}_1^1(G)$ the following assertions are valid for almost all $t \in (t_1, t_2)$:

a) if $\Gamma \subset {}_tG$ is a simple arc, then Γ is rectifiable,

b) the components of the set G_t are either points or homeomorphic images of a circle or of an open interval (in a countable number), and if $G_t^0 \subset G_t$ is the collection of point-components, then $H(G_t^0) = 0$.

Definition. Let B be a domain with compact closure $\overline{B} \subset G$. For any continuous mapping $T : G \to R^2$ set

$$l[T(B_t)] = \sum_k l[T(\beta_t^k)],$$

where β_t^k is the closure of a nonpoint component of the set B_t $(k = 1, 2, \ldots)$. By $l[T(G_t)]$ we denote $\lim_n l[T(B_t^n)]$, where $(B^n)_{n=1,2,\ldots}$ is some increasing sequence of domains with compact closure $\overline{B^n} \subset G$, exhausting the domain G.

This definition has meaning for almost all t, and $l[T(G_t)]$ does not depend on the choice of the sequence (B^n).

Theorem 1. If the mapping T belongs to the class $\widetilde{W}_p^1(G)$, $p \geq 1$, and the function $r(z) \in \widetilde{W}_q^1(G)$, $p^{-1} + q^{-1} = 1$, then for almost all $t \in (t_1, t_2)$ the components u, v of the mapping T are absolutely continuous inside G_t, i.e., on every simple arc $\Gamma \subset G_t$ the restrictions $u|_\Gamma$ and $v|_\Gamma$ are absolutely continuous as functions of the arc length H. Moreover,

$$l[T(G_t)] = \int_{G_t} |D_{sT}|\, dH.$$

Corollary. Suppose that, under the assumptions of Theorem 1, the function $|D_{sT}| \cdot |\nabla r| \in L_1(G)$. Then for almost all $t \in (t_1, t_2)$ the following assertion is true: if a nonpoint component $\Gamma \subset G_t$ is homeomorphic to the open interval $I = (0,1)$: $\Gamma = z(I)$, and $\alpha \in I$, then the finite limits

$$\lim_{\alpha\to 0} T(z(\alpha)) \quad \text{and} \quad \lim_{\alpha\to 1} T(z(\alpha))$$

exist.

This assertion is broader than certain existing analogues of the well-known theorem of P. Fatou (see ([7]), p. 66) on boundary values of analytic functions bounded inside a disk. For example, the results of ([8,9]) easily follow from it when $r(z) = \arg z$.

p. 2. Let $\lambda(z)$ and $\nu(z)$ be nonnegative measurable functions on G. It can be proved that if T and r are a mapping and a function of the class $\widetilde{W}_1^1(G)$, then almost everywhere on (t_1, t_2) the measurable functions

$$l_\lambda(G_t) = \int_{G_t} \lambda\, dH \quad \text{and} \quad l_\nu[T(G_t)] = \int_{G_t} \nu |D_{sT}|\, dH$$

are defined.

Below, by $\varphi(h)$ we denote a function defined for $h \geq 0$ and such that $\varphi(h) \geq 0$, $\varphi(0) = 0$, $\varphi''(h) > 0$.

Theorem 2. Let T and r be a mapping and a function of the class $\widetilde{W}_1^1(G)$, and let $g \subset (t_1, t_2)$ be a Borel set. If almost everywhere on g

$$0 < l_\lambda(G_t) < \infty, \tag{*}$$

then

$$\int_g l_\lambda(G_t) \cdot \varphi(l_\nu[T(G_t)]/l_\lambda(G_t))\, dt \leq \iint_{r^{-1}(g)} \lambda \cdot \varphi\left(\frac{\nu}{\lambda}|D_{sT}|\right) \cdot |\nabla r|\, dx\, dy.$$

This is a generalization of the well-known inequality underlying the investigations ([3,10]).

p. 3. In what follows, let $\lambda(z)$ and $\mu(w)$ be nonnegative $\mathfrak{B}$-functions given on $\overline{G}$ and $\overline{T(G)}$, respectively.

For any H-measurable set $B \subset R^2$ and $\mathfrak{B}$-function $\rho(z) \geqslant 0$, defined in R^2, we shall call the quantity

$$S_\rho(B) \stackrel{\text{def}}{=} \int_B \rho(z)\,dH$$

the ρ-length of the set B. (We note that the function ρ is H-measurable ([11,12]).)

A homeomorphic image $\pi \subset G$ of the half-open interval $I = [0,1)$ will be called a path (lying in G). Let $\pi = z(I)$, and $\pi_\alpha = z(I_\alpha)$, where z is a homeomorphism, and $I_\alpha = (\alpha, 1)$, $\alpha \in (0,1)$. The set

$$|\pi| = \bigcap_\alpha \pi_\alpha$$

will be called the cluster set of π. Let

$$|\Pi| \stackrel{\text{def}}{=} \bigcup_{\pi \in \Pi} |\pi|.$$

Let, further, K be a nondegenerate continuum lying inside G.

Definition. We shall say that a family Π of paths covers a set $A \subset G$, if:

1) $A \cap K = \emptyset$;
2) for every $\pi \in \Pi$ there exists an $\alpha \in (0,1)$ such that $\pi_\alpha \subset G \setminus G_K^A$,

where $G_K^A \supset K$ is the component of connectivity of the set $G \setminus A$.

Definition. A set Π of paths is called a 0^ρ-set (a null-ρ-set) if for every $\varepsilon > 0$ there exists a Borel set B covering Π with ρ-length $S_\rho(B) < \varepsilon$.

Let now $W \subset \widetilde{W}_q^1(G)$ be some family, and let Π be a certain family of paths (lying in G).

Definition. Π is called a $0_\varphi^\lambda(W)$-set if there exist a function $r \in W$ and a Borel set $g \subset (t_1, t_2)$ such that, for each $t \in g$, the set $G_t = \{z \in G \mid r(z) = t\}$ covers Π and condition $(*)$ is satisfied, and for any constant $c > 0$

$$\int_g l_\lambda(G_t) \cdot \varphi(c/l_\lambda(G_t))\,dt = \infty.$$

By the symbol $BL\varphi(\lambda, \mu, W)$ we shall denote the class of mappings $T \in \widetilde{W}_p^1(G)$ for which the integral

$$I(T,G) = \iint_G \lambda \cdot \varphi\left(\frac{\mu(T)}{\lambda}\,|D_{sT}|\right) \cdot |\nabla r|\,dx\,dy < \infty$$

for every function $r \in W$, $p^{-1} + q^{-1} = 1$. Obviously,

$$BL\varphi(\lambda, \mu, W) \subset BL\varphi(\lambda, \mu, W_1),$$

if $W_1 \subset W$.

If $p = 1$, $\varphi(h) = h^2$, $\lambda(z) = 1$, $\mu(w) = 1/(1 + |w|^2)$, and $W = \{|z - \zeta| \mid \zeta \in R^2\}$, then the class $BL\varphi(\lambda, \mu, W)$ includes the class $\widetilde{BL}$, studied in ([3]).

Theorem 3. *A homeomorphism $T : G \to \Delta$ of the class $BL\varphi(\lambda, \mu, W)$ carries every $0^\lambda_\varphi(W)$-set of paths Π (lying in G) into a 0^μ-set*

$$T(\Pi) = T(\pi)_{\pi \in \Pi}.$$

If the domain G is finitely connected, $\mu(w) = 1/(1 + |w|^2)$ and $S_\mu(\mathrm{Fr}\,\Delta) < \infty$, then $S_\mu(|T(\Pi)|) = 0$; in particular, for $\pi \in \Pi$ the cluster set of each path $T(\pi)$ is a point, i.e. there exists (finite or not)

$$\lim_{z \in \pi} T(z).$$

Here and below lim is to be understood conveniently in the sense of Moore–Smith; see ([13]).

For the formulation of one of the corollaries of this theorem we give the following

Definition. We shall call a point w of the extended plane R^2 a boundary value of the mapping $T : G \to R^2$, if there exists such a path $\pi \subset G$ that $|\pi| \subset \mathrm{Fr}\,G$ and

$$\lim_{z \in \pi} T(z) = w.$$

Denote by Ω the totality of all functions of the form $|w - \omega|$, where $\omega \in R^2$ is an arbitrary fixed point.

Corollary. *Let the function φ satisfy the condition*

$$\int_0^1 t\varphi(1/t)\, dt = \infty,$$

and $\mu(w) = 1/(1 + |w|^2)$. Then there does not exist a homeomorphism $T : G \to \Delta$, $T^{-1} \in BL\varphi(\mu, \lambda, \Omega)$, taking one and the same boundary value w on a set of Π paths (lying in G') which is not an O^λ-set.

From this there follows, for example, S. Agmon's result ([14]) on the nonconstancy of the boundary function on a nonzero set of the boundary of a disk under a quasiconformal mapping of the latter.

Remark. It is easy to see that the results given above extend almost verbatim to mappings of a plane domain into Euclidean n-space.

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REFERENCES

1. S. L. Sobolev, Some applications of functional analysis in mathematical physics, L., 1950.
2. V. I. Smirnov, Course of Higher Mathematics, 5, M., 1959.
3. G. D. Suvorov, Families of plane topological mappings, Novosibirsk, 1965.
4. B. P. Kufaev, DAN, 173, No. 6, 1257 (1967).
5. A. S. Kronrod, UMN, 5, issue 1 (35), 24 (1950).
6. I. Ya. Bakelman, A. L. Verner, UMN, 11, issue 1 (67), 173 (1956).
7. I. I. Privalov, Boundary Properties of Analytic Functions, M.–L., 1950.
8. K. S. Schild, DAN, 127, No. 2, 278 (1959).
9. S. L. Krushkal, Tr. Tomsk. Univ., Ser. Mech.-Math., 163, 54 (1963).
10. J. Lelong-Ferrand, *Representation conforme et transformations à intégrale de Dirichlet bornée*, Paris, 1955.
11. S. Saks, Theory of the Integral, M., 1949.
12. N. Dunford, J. T. Schwartz, Linear Operators, M., 1962.
13. K. Kuratowski, Topology, 1, M., 1966.
14. S. Agmon, J. Rat. Mech. Anal., 3, 763 (1954).

Note: Figure translations are in progress. See original paper for figures.