# CONTINUATION AND A FUNCTIONAL EQUATION FOR ZETA-FUNCTIONS WITH NON-ABELIAN CHARACTERS OF SIMPLE ALGEBRAS OVER NUMBER FIELDS

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## Abstract

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**MATHEMATICS**

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# CONTINUATION AND A FUNCTIONAL EQUATION FOR ZETA-FUNCTIONS WITH NON-ABELIAN CHARACTERS OF SIMPLE ALGEBRAS OVER NUMBER FIELDS

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1. Zeta-functions of algebras over number fields were first considered by H. Hey ([1,2]). He defined zeta-functions of algebras with division and proved their analytic continuation and a functional equation. Eichler ([3]) generalized these results to zeta-functions with abelian (i.e., one-dimensional) characters. Fujisaki ([4,5]) considered zeta-functions with abelian characters for simple algebras and, reducing them to zeta-functions of the corresponding algebras with division, proved for them continuation and a functional equation.

With the appearance of the theory of Hecke operators ([6]), the need arose to study zeta-functions connected with representations of the ring of Hecke operators on spaces of automorphic functions. This formulation of the question leads to the problem of studying zeta-functions with non-abelian characters, where the non-abelian characters that occur are positive-definite zonal spherical functions corresponding to automorphic functions. In general form this problem was posed in lectures of Godement ([7]), and there, by the Iwasawa–Tate method, was solved for algebras with division. Tamagawa ([8,9]) developed the theory of the Euler product for non-abelian zeta-functions. Kinoshita ([10]) proved continuation and a functional equation for non-abelian zeta-functions of the full matrix algebra over the field of rational numbers.

We have obtained a general result in this direction: analytic continuation and a functional equation have been proved for zeta-functions with non-abelian characters of an arbitrary simple central algebra over a field of algebraic numbers. We proceed to a detailed formulation of the results.

2. Let $A$ be a simple central algebra over a finite field of algebraic numbers $k$; $G$ the multiplicative group of $A$; $\mathfrak{D}$ a maximal order in $A$. For each valuation $\mathfrak{p}$ of the field $k$, denote by $k_\mathfrak{p}$ the completion of $k$ with respect to $\mathfrak{p}$; $A_\mathfrak{p} = A \otimes_k k_\mathfrak{p}$; $G_\mathfrak{p}$ the multiplicative group of $A_\mathfrak{p}$. For non-Archimedean

$\mathfrak{p}$, denote by $\mathfrak{D}_\mathfrak{p}$ the closure of $\mathfrak{D}$ in $A_\mathfrak{p}$, and by $U_\mathfrak{p}$ the subgroup generated by those $u_\mathfrak{p} \in G_\mathfrak{p}$ for which $u_\mathfrak{p}\mathfrak{D}_\mathfrak{p} = \mathfrak{D}_\mathfrak{p}$. For Archimedean $\mathfrak{p}$ put $U_\mathfrak{p} = \{u_\mathfrak{p} \in G_\mathfrak{p},\, uu_\mathfrak{p}^* = 1\}$, where $a_\mathfrak{p} \mapsto a_\mathfrak{p}^*$ is a positive involution of the algebra $A_\mathfrak{p}$. The subgroup $U_\mathfrak{p} \subset G_\mathfrak{p}$ is compact for all $\mathfrak{p}$; for non-Archimedean $\mathfrak{p}$ it is, in addition, open. For $x_\mathfrak{p} \in A_\mathfrak{p}$ put

$$V_\mathfrak{p}(x_\mathfrak{p}) = |Nx_\mathfrak{p}|_\mathfrak{p},$$

where $N$ is the norm in the regular representation of $A_\mathfrak{p}$ over $k_\mathfrak{p}$; $|\ |_\mathfrak{p}$ is the usual norm in the complete field $k_\mathfrak{p}$, for which the product formula ([11]) holds. Put, for $x_\mathfrak{p} \in A_\mathfrak{p}$,

$$\Phi_\mathfrak{p}(x_\mathfrak{p}) = \begin{cases} \exp(-\pi\ \mathrm{Tr}(x_\mathfrak{p} x_\mathfrak{p}^*)), & \text{if } \mathfrak{p} \text{ is Archimedean,} \\ \text{the characteristic function of the maximal order } \mathfrak{D}_\mathfrak{p} \subset A_\mathfrak{p}, & \text{if } \mathfrak{p} \text{ is non-Archimedean.} \end{cases}$$

where Tr is the reduced trace over the field $\mathbf{R}$ of real numbers; $x_\mathfrak{p} \mapsto x_\mathfrak{p}^*$ is a positive involution of $A_\mathfrak{p}$.

3. Let $J$ be the group of ideals of the group $G$, i.e. the restricted direct product ([11]) of the groups $G_\mathfrak{p}$ with respect to the subgroups $U_\mathfrak{p}^*$ over all valuations $\mathfrak{p}$ of the field $k$. Denote by $\Gamma \cong G$ the subgroup of principal ideals and put $U = \prod_\mathfrak{p} U_\mathfrak{p}$. For $g = (\dots g_\mathfrak{p} \dots) \in J$ put

$$V(g) = \prod_\mathfrak{p} V(g_\mathfrak{p}); \qquad \Phi(g) = \prod_\mathfrak{p} \Phi_\mathfrak{p}(g_\mathfrak{p}).$$

4. Let $H$ be a locally compact group; $V \subset H$ a compact subgroup. Denote by $L(H, V)$ the algebra of all complex continuous functions $f$ on $H$ with compact support which, for all $v, v' \in V$, $h \in H$, satisfy the condition $f(vhv') = f(h)$. Multiplication in $L(H, V)$ is defined as convolution:

$$(f * \varphi)(x) = \int_H f(xh^{-1})\varphi(h)\, dh, \qquad x \in H.$$

Recall that a continuous function $\omega$ on $H$ is called a (zonal) spherical function with respect to the subgroup $V$ ([8,9,12]) if for any $h, h' \in H$

$$\int_V \omega(hvh')\, dv = \omega(h)\omega(h'),$$

where $dv$ is the Haar measure on $V$, normalized by the condition $\int_V dv = 1$.

5. Let $\omega$ be a spherical function on $J$ with respect to $U$; then

$$\omega = \prod_\mathfrak{p} \omega_\mathfrak{p},$$

where $\omega_\mathfrak{p}$ are spherical functions on $G_\mathfrak{p}$ with respect to $U_\mathfrak{p}$ ([8]). For all $\mathfrak{p}$, $A_\mathfrak{p}$ is a simple algebra over $k_\mathfrak{p}$, so that $A_\mathfrak{p} = M_{r_\mathfrak{p}}(D_\mathfrak{p})$, where $D_\mathfrak{p}$ is a division algebra. The set of spherical functions on $G_\mathfrak{p}$ (with respect to $U_\mathfrak{p}$) is parametrized by $r_\mathfrak{p}$ complex parameters ([9]). We describe this parametrization for Archimedean $\mathfrak{p}$. Put

$$T_\mathfrak{p} = \{t = (t_{ij}) \in G_\mathfrak{p},\ t_{ij} \in D_\mathfrak{p},\ t_{ij} = 0 \text{ for } i > j,\ t_{ii} \in \mathbf{R},\ t_{ii} > 0\}.$$

Then $G_\mathfrak{p} = U_\mathfrak{p} T_\mathfrak{p}$. Let $(s(\mathfrak{p})) = (s_1(\mathfrak{p}), \dots, s_{r_\mathfrak{p}}(\mathfrak{p}))$ be arbitrary complex numbers. For $g_\mathfrak{p} \in G_\mathfrak{p}$, $g_\mathfrak{p} = u_\mathfrak{p} t_\mathfrak{p}$, where $u_\mathfrak{p} \in U_\mathfrak{p}$, $t_\mathfrak{p} \in T_\mathfrak{p}$, put

$$\alpha_{(s(\mathfrak{p}))}(g_\mathfrak{p}) = \alpha_{(s(\mathfrak{p}))}(t_\mathfrak{p}) = \prod_{i=1}^{r_\mathfrak{p}} t_{ii}^{\nu(-s_i(\mathfrak{p}) + (i-1))},$$

where $\nu = [D_\mathfrak{p} : \mathbf{R}]$. Then the function

$$\omega_{(s(\mathfrak{p}))}(g_\mathfrak{p}) = \int_{U_\mathfrak{p}} \alpha_{(s(\mathfrak{p}))}(g_\mathfrak{p}^{-1} u_\mathfrak{p})\, du_\mathfrak{p}$$

is a spherical function on $G_\mathfrak{p}$ with respect to $U_\mathfrak{p}$, and every spherical function on $G_\mathfrak{p}$ is obtained in the indicated way by a suitable choice of the parameters $s_i(\mathfrak{p})$. We shall call the parameters $s_i(\mathfrak{p})$ $(i = 1, \dots, r_\mathfrak{p})$ the roots of the spherical function $\omega_\mathfrak{p} = \omega_{(s(\mathfrak{p}))}$ and associate to each spherical function $\omega_\mathfrak{p}$ the polynomial

$$P_{\omega_\mathfrak{p}}(z) = \prod_{i=1}^{r_\mathfrak{p}} (r_\mathfrak{p} z - s_i(\mathfrak{p}))(r_\mathfrak{p} z - s_i(\mathfrak{p}) - 1),$$

where $s_i(\mathfrak{p})$ are the roots of $\omega_\mathfrak{p}$.

6. A continuous function $f$ on $J$ is called $\Gamma$-automorphic if:

I. $f(ug\gamma) = f(g)$ for all $u \in U,\ g \in J,\ \gamma \in \Gamma$.

II. For every function $\varphi \in L(J, U)$ there exists a complex number $\lambda_\varphi$ such that $\varphi * f = \lambda_\varphi \cdot f$.

To every nonzero $\Gamma$-automorphic function $f$ on $J$ there corresponds uniquely a spherical function $\omega$ on $J$ (with respect to $U$), which for all $g,\ g' \in J$ satisfies the condition

$$\int_U f(gug')\, du = \omega(g) f(g'). \tag{8}$$

We shall say that $\omega$ belongs to $f$. By the spectrum $s(\Gamma)$ of the discrete subgroup $\Gamma \subset J$ we shall mean the set of all spherical functions $\omega$ on $J$ with respect to $U$ that belong to some nonzero $\Gamma$-automorphic functions on $J$, are positive-definite functions on $J$, and for every $\xi$ in the center of $J$ and $g \in J$ satisfy the relation $\omega(\xi g) = \omega(g)$.

7. Let $\Gamma$ be the group of principal ideles, $\omega \in s(\Gamma)$. The zeta-function of the algebra $A$ with character $\omega$ is the function

$$\zeta(z,\omega) = \int_J \Phi(g)\omega(g^{-1})V(g)^z \, dg =$$

$$= \prod_{\mathfrak{p}} \int_{G_\mathfrak{p}} \Phi_\mathfrak{p}(g_\mathfrak{p})\omega_\mathfrak{p}(g_\mathfrak{p}^{-1})V_\mathfrak{p}(g_\mathfrak{p})^z \, dg_\mathfrak{p} = \prod_{\mathfrak{p}} \zeta_\mathfrak{p}(z,\omega_\mathfrak{p}),$$

where $dg = \prod_\mathfrak{p} dg_\mathfrak{p}$ is Haar measure on $J$, and the local measures $dg_\mathfrak{p}$ on $G_\mathfrak{p}$, for non-Archimedean $\mathfrak{p}$, are normalized by the condition

$$\int_{U_\mathfrak{p}} dg_\mathfrak{p} = 1.$$

The function $\zeta(z,\omega)$ is regular in the domain $\operatorname{Re} z > 1$. The $\mathfrak{p}$-factors $\zeta_\mathfrak{p}(z,\omega_\mathfrak{p})$ have been computed explicitly by Tamagawa (9); for Archimedean $\mathfrak{p}$ they are expressed in terms of the gamma-function, while for non-Archimedean $\mathfrak{p}$ they are rational functions of $p^{-z}$, where $p$ is the prime number divisible by $\mathfrak{p}$. Our main result is formulated as follows:

**Main theorem.** *In the notation and assumptions introduced above, the following assertions hold:*

I. *The function $\zeta(z,\omega)$ admits a meromorphic continuation to the whole $z$-plane.*

II. The function

$$\zeta(z,\omega) \prod_{\text{arch}} P_{\omega_\mathfrak{p}}(z)$$

is entire.*

III. *The function $\zeta(z,\omega)$ satisfies a functional equation of the form*

$$\zeta(z,\omega) = W(\omega)\Delta^{1/2-z}\zeta(1-z,\bar{\omega}),$$

*where $W(\omega)$ is a constant depending only on $\omega$; $|W(\omega)| = 1$; $\Delta$ is the absolute discriminant of the algebra $A$.*

The proof is based on combining the analysis of weight functions on the Archimedean components with the Poisson formula for the additive group of adeles of the algebra $A$.

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* This assertion remains valid if, instead of the product over all Archimedean $\mathfrak{p}$, one takes an arbitrary one of these polynomials $P_\omega(z)$.

*Note: Figure translations are in progress. See original paper for figures.*