ON EINSTEIN SPACES OF THE SECOND CLASS OF EMBEDDING

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Abstract

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ON EINSTEIN SPACES OF THE SECOND CLASS OF EMBEDDING

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1. Recently interest has revived in the problem of embedding a curved space-time in flat spaces of higher dimensions. In a series of articles under the general title "Seminar on the embedding problem" (^{1}) a survey is given of results on this problem and the possibilities of physical applications are discussed. These possibilities are hidden in the additional symmetries of the enveloping flat space S_m $(4 < m \leq 10)$, which can be used to obtain new conserved quantities. In article (^{1}) the embedding of many known V_4 of signature $(---+)$, satisfying Einstein' s field equations, into S_m is carried out; however, the class of these embeddings (defined as the number $p = m-4$, m being the least possible value) is not considered.

It is known that there exists no V_n of the first class with zero Ricci tensor (in the case $V_4(---+)$, the gravitational field equations in vacuum (^{2})). This result, obtained for a second quadratic form of simple type (^{3}), p. 240, turns out (^{4}) to hold for all possible characteristics of the second form $V_4(---+)$. In (^{4}) it is shown that the spaces of the first class satisfying the Einstein condition

$$R_{ij} = \chi g_{ij}, \qquad \chi = \text{const} \neq 0, \tag{1}$$

where g_{ij} is the metric tensor and R_{ij} is the Ricci tensor, are only spaces of constant curvature.

There is no general investigation of V_4 of the second class and higher, and only recently some necessary criteria were established for $V_4(---+)$ satisfying equations (1) with $\chi = 0$ to belong to the second class of embedding (^{5}).

In the present article, on the basis of an algebraic consideration of the embedding equations, invariant criteria are derived for Einstein spaces (1) of the second class.

2. Necessary and sufficient conditions for a V_n to belong to the second class of embedding consist (^{2}) in the fact that in this V_n there exist two symmetric tensors of the second quadratic forms $\underset{\tau}{h}_{ij}$ ($\tau = 1, 2$) and a vector Q_i satisfying the following equations:

$$R_{klij} = \underset{1}{e}\,\underset{1}{H}_{klij} + \underset{2}{e}\,\underset{2}{H}_{klij}, \qquad \underset{\tau}{H}^{kl}{}_{ij} = \underset{\tau}{h}^{k}{}_{[i}\underset{\tau}{h}^{l}{}_{j]}, \qquad \underset{1}{e} = \pm 1, \quad \underset{2}{e} = \pm 1, \tag{2}$$

$$\underset{1}{h}^{k}{}_{[i,j]} = -\underset{2}{e}\,\underset{2}{h}^{k}{}_{[i}Q_{j]}, \qquad \underset{2}{h}^{k}{}_{[i,j]} = \underset{1}{e}\,\underset{1}{h}^{k}{}_{[i}Q_{j]}, \tag{3}$$

$$Q_{[i,j]} = -\underset{1}{h}^{k}{}_{[i}\underset{2}{h}_{j]k}. \tag{4}$$

Here R_{klij} is the curvature tensor of V_n. The operation of alternation is applied without a numerical factor.

The tensors H_{klij} satisfy an identity which for $n = 4$, with the use of compound indices (^{6}) over skew-symmetric pairs of tensor indices, is written in the form

$$\varepsilon^{ab} H_{\tau ac} H_{\tau bd} = H_{\tau} \varepsilon_{ab} \quad (a,\ b,\ c,\ d = 1,\ 2, \dots, 6), \tag{5}$$

where $\varepsilon^{ab} \to \varepsilon^{klij}$ is the discriminant tensor of V_4, $H_\tau \equiv \det(h_{\tau ij}) / \det(g_{ij})$. Raising and lowering of bivector indices is carried out by means of the tensor $g_{ab} \to g_{klij} \equiv g_{ni} g_{lj} - g_{kj} g_{li}$. From identity (5) and equations (2) we find linear equations for $H_{\tau ab}$:

$$\varepsilon^{cd} \left(R_{ac} H_{\tau bd} + R_{bc} H_{\tau ad} \right) = e \left(K_{\tau ab} + G \varepsilon_{ab} \right), \tag{6}$$

where

$$K_{\tau ab} \equiv \varepsilon^{cd} R_{ac} R_{bd}, \qquad G = -G_1 \equiv H_1 - H_2.$$

Theorem 1. *The curvature tensor of every V_4 of the second embedding class satisfies the condition*

$$K \equiv \varepsilon^{ac} \varepsilon^{bd} R_{ab} K_{cd} = 0. \tag{7}$$

The proof follows directly from the algebraic properties of the tensors R_{ab} and $H_{\tau ab}$ and from relations (5) and (6).

Further investigation is based on consideration of the bivector $Q_{[i,j]} \equiv Q_{ij}$ from equality (4). Differentiating (3) with respect to l and alternating with respect to the indices ijl, we obtain:

$$R_{kp[ij}h^p_{1\,l]} = eQ_{2\,[ij}h_{1\,l]k}, \qquad R_{kp[ij}h^p_{2\,l]} = -eQ_{1\,[ij}h_{1\,l]k}. \tag{8}$$

Contracting (8) with respect to kl and taking into account the Einstein condition (1), we find

$$h_\tau Q_{ij} + Q_{\tau\,k[i}h^k_{\tau\,j]} = 0, \qquad h_\tau \equiv g^{ij}h_{\tau ij}. \tag{9}$$

Directly from (4) we have the equality

$$\varepsilon^{ab}Q_{1\,a}Q_{2\,b} = ee_{12}L, \qquad L \equiv g^{ab}K_{ab}. \tag{10}$$

From equalities (9), (10), (6), (7), (1), and (2) it follows:

Theorem 2. *The curvature tensor and the bivector Q_a of an Einstein space V_n of the second class satisfy the equations*

$$R_{ab}Q^b = \varkappa Q_a, \tag{11}$$

where $\varkappa = \mathrm{const}$ is real and is determined from (1).

Theorem 3. *The curvature tensor and the mean curvatures h_τ of Einstein spaces V_4 of the second class satisfy the condition*

$$(eh_1^2 + eh_2^2)L = 0. \tag{12}$$

If $Q_a \neq 0$, then the condition

$$\det(R_{ab} - \varkappa g_{ab}) = 0 \tag{13}$$

must be satisfied.

If $Q_a = 0$, then the vector Q_i is a gradient vector. In this case equalities (9) and (11) are satisfied identically and condition (13) does not apply. The curvature tensor and the tensors $h_{\tau ij}$ of such spaces must satisfy the conditions

$$L = 0, \qquad R_{kp[ij}h^p_{\tau\,l]} = 0, \qquad h^k_{1\,[i}h_{2\,j]k} = 0. \tag{14}$$

3. Let us apply the relations obtained to the study of Einstein spaces $V_4(---+)$ of three types according to Petrov’ s classification, writing them in the corresponding canonical frames (6).

Equality (7) imposes conditions on the stationary curvatures $k_s = \alpha_s + i\beta_s$ ($s = 1, 2, 3$). For $Q_a \neq 0$, there is an additional condition (13) on k_s. Solving (11) with allowance for (10), we find Q_a, from which h_{ij} is determined by (9), (8), and (6); moreover, the values found for Q_a and h_{ij} must satisfy equations (2) and (4). For $Q_a = 0$, equalities (14) give conditions on k_s and determine h_{ij}.

We give here invariant conditions on the stationary curvatures for the types of gravitational fields

Type I.
$\mathrm{rank}(Q_{ij}) = 4: \quad k_1 = -\varkappa, \quad k_2 = -k_3 = \alpha + i\beta, \quad \alpha \cdot \beta \neq 0.$
$\mathrm{rank}(Q_{ij}) = 2: \quad k_1 = -\varkappa, \quad k_2 = -k_3 = \alpha + i\beta, \quad \alpha \cdot \beta = 0.$

$$Q_{ij} = 0: \quad \begin{cases} 1) & k_1 = -2\alpha - \varkappa, \quad k_2 = k_3 = \alpha + i\beta, \\ 2) & k_s = \alpha_s. \end{cases}$$

Type II.
$\mathrm{rank}(Q_{ij}) = 4:$ the case is impossible.

$$\mathrm{rank}(Q_{ij}) = 2: \quad \begin{cases} 1) & k_1 = -\varkappa, \quad k_2 = 0, \\ 2) & k_1 = -k_2 = \varkappa, \\ 3) & k_s = 0. \end{cases}$$

$$Q_{ij} = 0: \quad \begin{cases} 1) & k_1 = \alpha_1, \quad k_2 = \alpha_2, \\ 2) & k_s = 0. \end{cases}$$

Type III.
$\mathrm{rank}(Q_{ij}) = 4:$ the case is impossible.
$\mathrm{rank}(Q_{ij}) = 2: \quad k_s = 0.$
$Q_{ij} = 0:$ there are no conditions on k_s.

In integrating equations (3), the following theorem will be useful.

Theorem 4. *If in V_n of the second class there exists a vector which is a common null direction of the tensors of the second quadratic forms, then this vector is covariantly constant and is determined up to a constant factor.*

Indeed, if such a vector v_i is covariantly constant, i.e. $v_{i,j} = 0$, then the first series of integrability conditions ((3), p. 87)

$$v_p R^p{}_{kij} = 0$$

is satisfied by virtue of equations (2) and the condition

$$h_{ij} v^j = 0.$$

The subsequent series of integrability conditions are satisfied by virtue of this condition and its differential consequences.

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Note: Figure translations are in progress. See original paper for figures.