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# GENERALIZED SYMMETRIC GAMMA- CORRELATION

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**Abstract**

**Full Text**

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## GENERALIZED SYMMETRIC GAMMA-CORRELATION

*(Presented by Academician S. N. Bernstein on 10 VI 1967)*

It is said that there is a symmetric gamma-correlation between nonnegative random variables  $x$  and  $y$  if these variables have the following joint distribution density:

$$p_1(x, y) = p(x)p(y) \left[ 1 + \sum_{k=1}^{\infty} R^k L_k^\alpha(x) L_k^\alpha(y) \right], \quad (1)$$

where  $\alpha > -1$ ,  $0 \leq R < 1$ , and  $L_k^\alpha(x)$  is the Laguerre polynomial defined by the formula

$$L_k^\alpha(x) = \sqrt{\frac{\Gamma(\alpha+1)\Gamma(\alpha+k+1)}{k!}} \sum_{r=0}^k \frac{(-1)^r \binom{k}{r} x^r}{\Gamma(\alpha+r+1)}. \quad (2)$$

The polynomials (2) are orthogonal and normal on the positive half-axis with weight equal to the marginal density

$$p(x) = \frac{1}{\Gamma(\alpha+1)} x^\alpha e^{-x}. \quad (3)$$

The nonnegativity of (1) in the first quadrant follows from the Miller-Lebedev formula <sup>(1)</sup>

$$p_1(x, y) = \frac{1}{(1-R)\Gamma(1+\alpha)} \left(\frac{xy}{R}\right)^{\alpha/2} e^{-(x+y)/(1-R)} I_\alpha \left(2\sqrt{\frac{xyR}{1-R}}\right), \quad (4)$$

where  $I_\alpha(z)$  is the Bessel function of imaginary argument.

**Remark 1.** The correlation coefficient  $R$  is always considered nonnegative; negativity of  $R$  would mean that  $x$  and  $y$  have different signs, and this leads to the loss of symmetry and lies outside the scope of the questions considered here.

**Definition.** If the joint distribution density of nonnegative  $x$  and  $y$  has the form

$$p(x, y) = p(x)p(y) \left[ 1 + \sum_{k=1}^{\infty} c_k L_k^\alpha(x) L_k^\alpha(y) \right], \quad (5)$$

where  $\{c_k\}$  is some sequence of nonnegative numbers satisfying the condition

$$\sum_{k=1}^{\infty} c_k^2 < \infty, \quad (6)$$

which ensures convergence of the series (5) in the mean, then it is said that  $x$  and  $y$  are connected by a generalized symmetric gamma-correlation.

**Theorem.** *In order that the sum of the series (5) be nonnegative for  $x \geq 0$  and  $y \geq 0$ , it is necessary and sufficient that the sequence  $\{c_k\}$  be the moment sequence of some probability distribution concentrated on the half-interval  $[0, 1)$ .*

**Necessity.** Suppose that the sum of the series (5) is nonnegative; then it is a distribution density, and  $c_k$  are the correlation coefficients between  $L_k^\alpha(x)$  and  $L_k^\alpha(y)$ ,  $k = 1, 2, \dots$ , therefore

$$0 \leq c_k < 1. \quad (7)$$

Noting that the Fourier transform of the weighted Laguerre polynomial has the form

$$\frac{1}{\Gamma(\alpha + 1)} \int_0^\infty L_k^\alpha(y) y^\alpha e^{-y+ity} dy = \sqrt{\frac{\Gamma(\alpha + k + 1)}{k! \Gamma(\alpha + 1)}} \frac{(-it)^k}{(1 - it)^{\alpha+k+1}}, \quad (8)$$

we find the conditional characteristic function of the random variable  $y/x$  for fixed  $x \neq 0$

$$\begin{aligned} \varphi_x \left( \frac{t}{x} \right) &= M_x e^{i \frac{t}{x} y} = \int_0^\infty e^{i \frac{t}{x} y} p(y) \left[ 1 + \sum_{k=1}^{\infty} c_k L_k^\alpha(x) L_k^\alpha(y) \right] dy = \\ &= \frac{1}{(1 - it/x)^{1+\alpha}} \left[ 1 + \sum_{k=1}^{\infty} \frac{(-it)^k}{k!} c_k \frac{\Gamma(\alpha + k + 1)}{(x - it)^k} \sum_{r=0}^k \frac{(-1)^r \binom{k}{r}}{\Gamma(\alpha + r + 1)} x^r \right]. \quad (9) \end{aligned}$$

By means of elementary identical transformations, we represent the expression in square brackets in (9) as the sum of the following three terms:

$$\omega(t) = 1 + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} c_k,$$

$$J_1 = \sum_{k=1}^{\infty} \frac{(it)^k}{k!} c_k \left[ \frac{1}{(1-it/x)^k} - 1 \right], \quad (10)$$

$$J_2 = \sum_{k=1}^{\infty} \frac{(it)^k}{k!} c_k \frac{1}{(1-it/x)^k} \sum_{\nu=1}^k \frac{(-1)^\nu \binom{k}{\nu} \Gamma(\alpha + k + 1)}{x^\nu \Gamma(\alpha + k + 1 - \nu)}$$

and show that, for sufficiently large  $x$ , the sums  $J_1$  and  $J_2$  are arbitrarily small in absolute value.

Noting that for  $\beta > 0$ ,  $|t| < T$ , and  $x > 2T$

$$\left| \frac{1}{(1-it/x)^\beta} - 1 \right| < \frac{T}{x} \beta \quad (11)$$

and taking (7) into account, we obtain

$$|J_1| < \frac{1}{x} \sum_{k=1}^{\infty} \frac{T^k}{k!} T^k = \frac{T^2}{x} \sum_{k=1}^{\infty} \frac{T^{k-1}}{(k-1)!} = \frac{T^2}{x} e^T. \quad (12)$$

To obtain an estimate of  $|J_2|$ , note that

$$\frac{1}{k!} \binom{k}{\nu} \frac{\Gamma(\alpha + k + 1)}{\Gamma(\alpha + k + 1 - \nu)} < \frac{\binom{[\alpha]+1+k}{\nu}}{(k-\nu)!} < \frac{2^{[\alpha]+1+k}}{(k-\nu)!}, \quad (13)$$

and since, by virtue of (7),  $|c_k| < 1$ , and, moreover,  $1/|1-it/x|^k < 1$ , for  $|t| < T$

$$|J_2| < \sum_{k=1}^{\infty} T^k 2^{[\alpha]+1+k} \sum_{\nu=1}^k \frac{1}{x^\nu (k-\nu)!}. \quad (14)$$

We note that

$$\begin{aligned} \sum_{\nu=1}^k \frac{1}{x^\nu (k-\nu)!} &= \sum_{\nu=1}^{[k/2]} \frac{1}{x^\nu (k-\nu)!} + \sum_{[k/2]+1}^k \frac{1}{x^\nu (k-\nu)!} < \\ &< \frac{1}{x} \frac{[k/2]}{[k/2]!} + \frac{[k/2]}{x^{1+[k/2]}} < \frac{2^k}{k} \left[ \frac{1}{[k/2]!} + \frac{1}{x^{[k/2]}} \right]. \end{aligned} \quad (15)$$

Therefore

$$\begin{aligned}
 |J_2| &< \frac{2^{[\alpha]+1}}{x} \left( \sum_{k=1}^{\infty} \frac{(4T)^k}{[k/2]!} + \sum_{k=1}^{\infty} \frac{(4T)^k}{x^{[k/2]}} \right) < \\
 &< \frac{2^{[\alpha]+1}}{x} \left[ (1+4T)e^{16T^2} + \frac{4T}{\sqrt{x}(1-4T/\sqrt{x})} \right] < \frac{2^{[\alpha]+1}}{x} [(1+4T)e^{16T^2} + 1],
 \end{aligned} \tag{16}$$

if  $x > 64T^2$  and  $|t| < T$ .

Finally, note that (11), for  $\beta = 1 + \alpha$ , takes the form

$$\left| \frac{1}{(1-it/x)^{1+\alpha}} - 1 \right| < (1+\alpha) \frac{T}{x}, \tag{11'}$$

and

$$|\omega(t)| \leq \sum_{k=0}^{\infty} \left| \frac{(it)^k}{k!} c_k \right| < e^T. \tag{17}$$

After this, represent  $\varphi_x(t/x)$  in the form

$$\begin{aligned}
 \varphi_x \left( \frac{t}{x} \right) &= \left[ 1 + \frac{1}{(1-it/x)^{1+\alpha}} - 1 \right] (\omega(t) + J_1 + J_2) \\
 &= \omega(t) + \omega(t) \left[ \frac{1}{(1-it/x)^{1+\alpha}} - 1 \right] + \frac{J_1 + J_2}{(1-it/x)^{1+\alpha}},
 \end{aligned}$$

therefore

$$\left| \varphi_x \left( \frac{t}{x} \right) - \omega(t) \right| < |\omega(t)| \left| \frac{1}{(1-it/x)^{1+\alpha}} - 1 \right| + |J_1| + |J_2|. \tag{18}$$

Now let  $\varepsilon > 0$  be arbitrarily small; then, on the basis of (12), (16), (11'), and (17), taking  $x$  greater than each of the numbers  $64T^2$  and

$$\frac{1}{\varepsilon} \left[ e^T(1+\alpha)T + T^2e^T + 2^{[\alpha]+1}(1+4T)e^{16T^2} + 1 \right],$$

we obtain

$$|\varphi_x(t/x) - \omega(t)| < \varepsilon \tag{19}$$

for all  $|t| < T$ .

Thus,  $\omega(t)$  is a characteristic function as the limit of the sequence of characteristic functions  $\varphi_x(t/x)$  as  $x \rightarrow \infty$ , and, consequently,  $\{c_k\}$  is a moment sequence, since  $c_k = \omega^{(k)}(0)/i^k$ .

Moreover, since, according to (6),  $c_k \rightarrow 0$  as  $k \rightarrow \infty$ , the random variable generating the sequence  $\{c_k\}$  cannot, with positive probability, exceed one in absolute value or take the value equal to one with positive probability. Negative values of this variable have been excluded from consideration by us in view of Remark 1 made at the beginning of this article. Thus,  $\{c_k\}$  is the moment sequence of some random variable concentrated on the half-interval  $[0, 1)$ .

**Sufficiency.** Let  $\{c_k\}$  form the sequence of moments of some random variable  $\xi$ , concentrated on the half-interval  $[0, 1)$  and given by the distribution function  $F(\xi)$ . For each fixed  $\xi$  consider expression (1), putting in it  $R = \xi$ ; integrating this expression with respect to  $\xi$  with the integral weight  $dF(\xi)$ , which is legitimate for series converging in the mean, we obtain, according to (4) and (5),

$$p(x, y) = \int_0^1 \left(\frac{xy}{\xi}\right)^{\alpha/2} \frac{1}{1-\xi} e^{-(x+y)/(1-\xi)} I_\alpha \left(2\frac{\sqrt{xy\xi}}{1-\xi}\right) dF(\xi) \geq 0. \quad (20)$$

Expression (20) is nonnegative as the integral of a nonnegative function, i.e. (5) in this case is indeed a density.

**Remark 2.** For  $\alpha = -\frac{1}{2}$ , replacing  $x$  and  $y$  by  $x^2$  and  $y^2$  leads to bilinear expansions of a two-dimensional density in Hermite polynomials of even order. Thus, from the theorem proved there follows the corresponding result obtained directly in (2).

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## REFERENCES

<sup>1</sup> W. Myller-Lebedeff, *Math. Ann.*, **64**, 388 (1907). <sup>2</sup> O. V. Sarmanov, DAN, **168**, No. 1, 32 (1966).

*Note: Figure translations are in progress. See original paper for figures.*

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