VACUUM MAGNETIC MOMENT OF AN ELECTRON MOVING IN A CONSTANT AND HOMOGENEOUS MAGNETIC FIELD

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Abstract

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VACUUM MAGNETIC MOMENT OF AN ELECTRON MOVING IN A CONSTANT AND HOMOGENEOUS MAGNETIC FIELD

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As is well known, the theoretical explanation of the nature of the anomalous magnetic moment of the electron and the calculation of its magnitude were first carried out by Schwinger (^{1}). Considering the part of the energy of the vacuum interaction of the electron that arises owing to the presence of an external magnetic field, Schwinger showed that, in the nonrelativistic approximation, the terms linear in the magnetic-field strength vector lead to a change in the electron g-factor. The calculations, first performed (^{1}) with accuracy up to terms of order $\alpha = e^2/\hbar c$, were subsequently continued ($^{2-4}$) up to quantities of order α^3. It then turned out that an electron behaves in a constant external magnetic field as if it possessed a static magnetic moment

$$\mu = \mu_0(1 + \alpha/2\pi - 0.328\alpha^2/\pi^2 + 0.13\alpha^3/\pi^3)$$

$$(\mu_0 = -e_0\hbar/2mc \text{ –the Bohr magneton}).$$

In connection with the great capabilities of highly developed precision experimental measurement techniques (^{5}), a more complete investigation of the energy of the vacuum interaction of the electron is of interest. Taking into account higher terms in the field strength (to first order in α) has shown (^{6}) that the magnitude of the magnetic moment is a rather complicated function of the field.

Let us further note that in all the works listed above the nonrelativistic approximation to the problem was considered. This is especially clearly evident in work (^{6}), where the electron is taken to be at rest. In addition, it was assumed that the magnetic field is weak: the field strength H is bounded in its variation by the limit

$$H \ll H_0 = m^2c^3/e_0\hbar = 4.41 \cdot 10^{13} \text{ oersted.}$$

In the present work we wish to carry out an analysis of the magnitude of the magnetic moment of the electron to first order in α, free of the indicated restrictions. In doing so we shall follow the method proposed by Luttinger (^{7}) and developed for relativistic electrons in (^{8}).

The Dirac equation, taking into account the interaction of the electron with the electromagnetic vacuum, can be taken in the form (^{9})

$$(\gamma_\mu P_\mu - m)\psi(x_\mu) = \int M(x_\mu, y_\mu)\psi(y_\mu)d^4y,$$

where $\gamma_\mu = -i\rho_3\alpha_\mu$, $\alpha_\mu = (\vec{\alpha}, iI)$ are the Dirac matrices; the kinetic momentum

$$P_\mu = i\partial/\partial x_\mu + e_0(A_\mu + A_\mu^{\text{vac}}) \quad (c = \hbar = 1),$$

and the mass operator in first order in the constant α

$$M(x, y) = ie_0^2 \sum_{k,m} \gamma^k S^c(x, y)\gamma^m D^c_{km}(y - x).$$

contains the Green’ s function of the Dirac equation with allowance for the external magnetic field $S^c(x, y)$ and the Green’ s function of the photon field D^c_{km}.

For the case of a constant and homogeneous external magnetic field (directed along the z-axis), the stationary Dirac equation with allowance for radiative corrections is reduced to the form (^{8,10})

$$(E - \mathscr{H})\psi(\mathbf{r}) = \hat{R}\psi(\mathbf{r}) = \int K(\mathbf{r}, \mathbf{r}')\psi(\mathbf{r}')\, d^3r',$$

where $E = c\hbar K$ is the electron energy, and the operator $\mathscr{H}$ contains only the external magnetic field. The kernel K on the right-hand side of this equation is the sum

$$K(\mathbf{r}, \mathbf{r}') = \frac{e_0^2}{4\pi^2} \sum_{n',\varepsilon} \int \frac{d^3\chi}{\chi} \frac{e^{i\chi\mathbf{r}}\alpha_i\psi_{n'}(\mathbf{r})\psi^+_{n'}(\mathbf{r}')\alpha_\mu e^{-i\chi\mathbf{r}'}}{K_n - \varepsilon(K_{n'} + \chi)}$$

over all intermediate states n', including the electron ($\varepsilon = 1$) and the positron ($\varepsilon = -1$).

The calculation of the matrix elements of the operator $\hat{R}$, which determine the radiative corrections to the electron energy, can be carried out with the aid of the known (see, for example, (^{10})) exact wave functions of an electron moving in a magnetic field. We shall assume that the electron has no motion along the field. Then the electron energy is connected only with the principal quantum number $n = 0, 1, 2, \ldots$

$$K_n = \sqrt{k_0^2 + 2\gamma n}, \qquad \gamma = e_0 H/c\hbar, \qquad k_0 = mc/\hbar.$$

Let us first consider the case of excited electron states, when the principal quantum number $n \neq 0$. Then, representing the expression for the energy of the vacuum interaction directly connected with the vacuum moment in the form *

$$W^{\text{vac}} = -\mu H, \qquad \mu = -\frac{\alpha}{2\pi}\mu_0 f(n,a),$$

for the function $f(n,a)$ we obtain the expression

$$f(n,a) = -8a \sum_{n'=0}^{\infty} \int_0^{\infty} \int_0^{\pi} \frac{x\,dx\,\sin\theta\,d\theta}{\left(\sqrt{\xi + x^2\cos^2\theta} + x\right)^2 - 1} \times \left[1 + \frac{\xi - 1 + x^2\sin^2\theta}{x\sin^2\theta\sqrt{\xi + x^2\cos^2\theta}}\right]\left[I_{nn'}^2(z) - I_{n-1,n'-1}^2(z)\right]. \tag{1}$$

Here

$$a = k_0^2/2\gamma = H_0/2H; \qquad \xi = (n' + a)/(n + a); \qquad z = (n + a)x^2\sin^2\theta;$$

$$I_{nn'}(z) = \sqrt{n'!/n!}\, e^{-z/2} z^{(n-n')/2} L_n^{n-n'}(z)$$

is the Laguerre function. We note that the expression contains no divergences (see also (^{8})).

The case in which the electron is at rest ($n = 0$) is in a certain sense special, since in this state the electron spin is strictly oriented opposite to the direction of the magnetic field. For this case we have

$$f(0,a) = -8a \sum_{n'=0}^{\infty} \int_0^{\infty} \int_0^{\pi} \frac{x\,dx\,\sin\theta\,d\theta}{\left(\sqrt{\xi + x^2\cos^2\theta} + x\right)^2 - 1} \times \left[\frac{x}{\sqrt{\xi + x^2\cos^2\theta}} I_{0,n'-1}^2(z') + \left(2 + \frac{2x^2\sin^2\theta + \xi - 1}{x\sin^2\theta\sqrt{\xi + x^2\cos^2\theta}}\right) I_{0,n'}^2(z')\right], \tag{2}$$

$$\xi = 1 + n'/a, \qquad z' = ax^2 \sin^2\theta.$$

This expression contains a divergence not connected with the magnetic field and removed by mass renormalization.

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* Such a notation follows from considerations of relativistic covariance (see (^{11})), according to which the operator of the anomalous magnetic moment must have the form

$$\hat{R}^v = \mu\rho(\sigma\mathbf{H}).$$

A. **The case of a weak magnetic field** ($H \ll H_0$). In the case of a weak magnetic field one may assume that $n + a \gg a \geq 1$ and expand expressions (1) and (2) in powers of $(n + a)^{-1}$. Then we obtain

$$f(n, a) = 1 - \frac{7}{3a^2}\left(\ln a - \frac{576\ln 2 - 83}{420}\right), \tag{3}$$

$$f(0, a) = 1 - \frac{1}{a}\left(\frac{4}{3}\ln a - \frac{13}{18}\right). \tag{4}$$

The expression for an electron at rest (11) differs from the result known in the literature (^{6}) by the numerical coefficient 13/18 instead of 47/90. Comparison of formulas (3) and (4) shows that the nonlinear corrections to the value of the anomalous magnetic moment in the ground state ($n = 0$) differ substantially from the corrections for excited states ($n \neq 0$).

It is also characteristic that expression (3) does not depend on the electron energy. However, this result is valid only under the assumption that $n \ll a$. A more refined analysis of expression (1) shows that formula (3) is applicable only for the case when $n \ll a^3$, i.e., up to electron energies $E < E_{\mathrm{cr}} = mc^2 H_0/H$*. In the other limiting case, when $n \gg a^3$, the anomalous moment decreases sharply with increasing energy:

$$f(n, a) = \frac{a\Gamma(1/3)}{\sqrt[3]{18n}} = \frac{3^{1/3}\Gamma(1/3)}{6}\left(\frac{E}{mc^2}\frac{H}{H_0}\right)^{-2/3}.$$

B. **The case of a strong magnetic field** ($H > H_0$). Analyzing expressions (1) and (2) in the case of a strong magnetic field, one can verify that as $a \to 0$, of the entire sum only the term $n' = 0$ proves to be most significant. In this case, in expression (2) the divergence due to mass renormalization must be removed. Then the asymptotic expression for the function has the form

$$f(n,a) = \begin{cases} 2a \ln a/n, & n \neq 0, \\ -2a(\ln^2 a + 2c \ln a), & n = 0 \end{cases}$$

(c is Euler’ s constant).

Thus, to first order in the constant α, the value of the electron magnetic moment is a complicated function of the field and of the electron energy. As the magnitude of the field changes, this function may change sign.

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named after M. V. Lomonosov

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[*] Let us note that an analogous criterion appears as the limit of applicability of the classical theory of synchrotron radiation; see ([12]).

Note: Figure translations are in progress. See original paper for figures.

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