AUTOMATON MAPPINGS AND FUNCTIONAL SEMIMATRICES OVER THE SYMMETRIC SEMIGROUP OF AN ALPHABET

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Abstract

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AUTOMATON MAPPINGS AND FUNCTIONAL SEMIMATRICES OVER THE SYMMETRIC SEMIGROUP OF AN ALPHABET

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1. A semimatrix of order k over a set M $(k = 1, 2, \dots, \infty)$ is a table of the form $(\varphi_{ij})_{j \le i}$ $(i, j = 0, 1, \dots, k)$, where $\varphi_{ij} \in M$, or, in other words, a function of two natural variables i and j with values in M, defined only on those pairs (i, j) for which $j \le i$.

Let us take as M the totality of all possible functions $\alpha(x)$, defined on a fixed finite alphabet X of $n \ge 2$ elements, with values in its symmetric semigroup $S'(X)$. A semimatrix $\alpha_{ij}(x)_{j \le i}$ over such a set will be called a functional semimatrix over $S'(X)$; moreover, by definition we require that its diagonal elements be constant functions: $\alpha_{ii}(x) = \alpha_{ii} \in S'(X)$, $x \in X$. Two functional semimatrices (f.s.m.) $(\alpha_{ij}(x))$ and $(\beta_{ij}(x))$ over $S'(X)$ are called equivalent if, for any $i = 0, 1, \dots$ and every vector $\mathbf{x} = (x_0, \dots, x_{i-1}) \in X^i$, one has

$$\alpha_{i0}(x_0) \dots \alpha_{i(i-1)}(x_{i-1})\alpha_{ii} = \beta_{i0}(x_0) \dots \beta_{i(i-1)}(x_{i-1})\beta_{ii}.$$

This relation is reflexive, symmetric, and transitive.

By the derivative $(\alpha_{ij}(x))'_a$ of an f.s.m. $(\alpha_{ij}(x))$ with respect to the vector $\mathbf{a} = (a_0, \dots, a_{s-1}) \in X^s$, $s < \infty$, we mean the f.s.m. $(\beta_{ij}(x))$ for which

$$\beta_{ij}(x) = \begin{cases} \alpha_{(s+i)0}(a_0) \dots \alpha_{(s+i)(s-1)}(a_{s-1})\alpha_{(s+i)s}(x), & \text{if } j = 0; \\ \alpha_{(s+i)(s+j)}(x), & \text{if } j > 0. \end{cases}$$

A semimatrix of infinite order over an arbitrary set will be called periodic if it has periodic (more precisely, ultimately periodic) sequences of columns and each column.

Proposition 1. If the f.s.m. $(\alpha_{ij}(x))$ and the f.s.m. $(\beta_{ij}(x))$ over $S'(X)$ are equivalent, then, for any $\mathbf{a} \in X^s$, $s < \infty$, the f.s.m. $(\alpha_{ij}(x))'_a$ and $(\beta_{ij}(x))'_a$ are also equivalent.

Proposition 2. If the f.s.m. $(\alpha_{ij}(x))$ is periodic, then $(\alpha_{ij}(x))'_a$ is also periodic for any $\mathbf{a} \in X^s$, $s < \infty$.

To the f.s.m. $(\alpha_{ij}(x))$ we associate a mapping α of the set X^∞ into itself, which carries the vector $\mathbf{x} = (x_0, \dots, x_r, \dots) \in X^\infty$ into $\mathbf{x}' = (x'_0, \dots, x'_r, \dots)$, where

$$x'_r = \alpha_{r0}(x_0) \dots \alpha_{r(r-1)}(x_{r-1})\alpha_{rr}x_r \quad (r = 0, 1, \dots)$$

(φx denotes the result of applying the mapping φ from $S'(X)$ to the letter x from X).

For the formulation of the following proposition we recall a theorem on the connection between automaton mappings and wreath products of symmetric groups and semigroups. Let semigroups $P_0, P_1, \dots$ of mappings of the sets $X_0, X_1, \dots$ be given. Consider a table of functions

$$\alpha = [\alpha_0, \alpha_1(x_0), \dots, \alpha_r(x_0, \dots, x_{r-1}), \dots], \tag{1}$$

where $\alpha_r(x_0, \dots, x_{r-1})$ is defined on the set $X_0 \times \dots \times X_{r-1}$ with values in P_r. To each such table there corresponds a mapping α of the set $X_0 \times X_1 \times \dots$ into itself, carrying $\mathbf{x} = (x_0, \dots, x_r, \dots)$ into $\mathbf{x}' = (x'_0, \dots, x'_r, \dots)$, where $x'_r = \alpha_r(x_0, \dots, x_{r-1})x_r$. All such mappings form a semigroup, which is called the wreath product of the semigroups $P_0, P_1, \dots$. This construction was first studied by L. A. Kaluzhnin (see, for example, ([3])) for

groups. Put $X_0 = X_1 = \cdots = X$; $P_0 = P_1 = \cdots = S'(X)$. The semigroup obtained, i.e., the countable wreath product of the symmetric semigroups $S'(X)$, coincides with the semigroup of all automaton mappings over the alphabet X. This result is an obvious generalization of the author' s theorem on the representation of the group of automaton permutations in the form of a countable wreath product of symmetric groups $S(X)$ (see ([4],[5])).

Proposition 3. The mapping a, induced by the functional semimatrix $(\alpha_{ij}(x))$ over $S'(X)$, is an automaton mapping whose representation functions (1) admit decompositions of the form

$$\alpha_r(x_0, \dots, x_{r-1}) = \alpha_{r0}(x_0)\alpha_{r1}(x_1)\cdots\alpha_{r(r-1)}(X_{r-1})\alpha_{rr} \qquad (r = 0, 1, \dots).$$

Proposition 4. Two f.s. are equivalent if and only if they induce the same automaton mapping.

Proposition 5. If an f.s. $(\alpha_{ij}(x))$ induces an automaton a, then $(\alpha_{ij}(x))'_a$ induces its state a_a.

Proposition 6. If M is an arbitrary set of automaton mappings, and M' is the set of states of the mappings belonging to it, then the correspondence $M \to M'$ is a closure (see ([6])). The set of automaton mappings induced by f.s. is closed.

From these propositions and Raney' s theorem it follows that

Theorem 1. Let, in the representation (1) of an automaton mapping a over the alphabet X, the functions admit decompositions of the form (2). Fix one such decomposition for each r, and construct an automaton whose states are the functional semimatrices $(\alpha_{ij}(x))'_a$, and whose transition and output functions are given by the formulas

$$\delta[(\alpha_{ij}(x))'_a, y] = (\alpha_{ij}(x))'_{ay}, \qquad \lambda[(\alpha_{ij}(x))'_a, y] = (\alpha_{ij}(x))'_a y;$$

as the initial state we choose $(\alpha_{ij}(x))$. The automaton thus constructed is the minimal automaton inducing the mapping a.

If, moreover, the f.s. $(\alpha_{ij}(x))$ is periodic, then the mapping a is finite-automaton, and the constructed minimal automaton has no more than

$$(k+m)n^{n(\chi+\mu)}$$

states, where k, m are the numbers of elements before the first period and in the period of the sequence of columns; χ_j, μ_j are the corresponding numbers for the j-th column,

$$\chi = \max(\chi_0, \dots, \chi_{k+m-1}), \qquad \mu = \mathrm{l.\,c.\,m.}(\mu_0, \dots, \mu_{k+m-1}).$$

The set of automaton mappings specified by functional semimatrices is closed.

2. Let us narrow the class of admissible functional semimatrices. We shall call an f.s. $(\alpha_{ij}(x))$ completely normalized if two conditions are satisfied: a) for any $x \in X$ and $i, j = 0, 1, \ldots$, the elements $\alpha_{ij}(x)$ are invertible mappings from $S'(X)$; b) the subgroups A_{ij} of the semigroup $S'(X)$, generated by the subsets $\alpha_{ij}(X) \subseteq S'(X)$ $(i \neq j)$, with common index i and distinct indices j, are pairwise elementwise permutable: from $j_1 \neq j_2$ and $\alpha_1 \in A_{ij_1}$, $\alpha_2 \in A_{ij_2}$ it follows that $\alpha_1\alpha_2 = \alpha_2\alpha_1$. We shall call a function $a(x)$ homogeneous with respect to $a \in X$ if $a(a) = \varepsilon$, where ε is the identity of the semigroup $S'(X)$. It is agreed to say that a completely normalized f.s. $(\alpha_{ij}(x))$ has normal form with respect to a if its off-diagonal elements are homogeneous with respect to a.

Proposition 7. For every completely normalized f.s. and every $a \in X$, there exists a unique f.s. equivalent to it, in normal form with respect to a.

Proposition 8. If an f.s. is completely normalized, then any derivative of it is completely normalized.

Theorem 2. In order that the automaton mapping induced by a completely normalized f.s. in normal form with respect to a be finite-automaton, it is necessary and sufficient that this f.s. be periodic. The set of all automaton mappings induced by completely normalized f.s. is closed.

3. Let a binary alphabet $X = \langle x, y \rangle$ be given. Denote

$$\varepsilon = \begin{pmatrix} xy \\ xy \end{pmatrix}, \quad \sigma = \begin{pmatrix} xy \\ yx \end{pmatrix}, \quad \xi = \begin{pmatrix} xy \\ xx \end{pmatrix}, \quad \eta = \begin{pmatrix} xy \\ yy \end{pmatrix}.$$

If $a(x)$ is a function on X with values in $S'(X) = \langle\varepsilon, \sigma, \xi, \eta\rangle$, then by $Q_\zeta^{a(x)}$ ($\zeta \in S'(X)$) we denote the set of those letters z from X for which $a(z) = \zeta$. Thus, between the functions $a(x)$ and partitions of the alphabet X into four disjoint subsets $Q_\varepsilon^{a(x)}, Q_\sigma^{a(x)}, Q_\xi^{a(x)}, Q_\eta^{a(x)}$ (some of which may be empty), a one-to-one correspondence is established. Denote by P_z^ζ ($\zeta \in S'(X), z \in X$) the full preimage of the letter z under the mapping ζ of the alphabet X into itself.

The following lemma is important; for its formulation we introduce one definition. If an infinite sequence, beginning with the k-th term, is divided into segments of length s, and the latter are numbered by the numbers $I = 0, 1, \ldots$, while inside each I-th segment the elements are numbered by the numbers $i = 0, 1, \ldots, s-1$, then the pair of numbers (i, I) corresponding to any element of the sequence, beginning with the k-th term, will be called the (k, s)-number of this element, and we shall say that a (k, s)-numbering of the elements of the sequence has been performed.

Lemma 1 (on periodic semimatrices). *In order that a semimatrix $(\varphi_{ij})_{j\le i}$ $(i, j = 0, 1, \ldots)$ over an arbitrary set M be periodic, it is necessary and sufficient that there exist such a $(q+r, s)$-numbering of the sequence of its rows under which the row $[a_{i,I}]$ with $(q+r, s)$-number (i, I) has the form*

$$a_{(i,I)} = \vec{\alpha}_i(\vec{\beta}_i)^I(\vec{\beta}_i|^{i+1}\vec{\gamma}_i \quad (I = 0, 1, \ldots;\ i = 0, 1, \ldots, s-1),$$

where $\vec{\alpha}_i, \vec{\beta}_i, \vec{\gamma}_i$ are finite-dimensional vectors over M, independent of I, of dimensions r, s, q, respectively, independent of i; $(\vec{\beta}_i)^I$ is the result of I-fold appending of the vector $\vec{\beta}_i$, and $(\vec{\beta}_i|^{i+1}$ denotes the initial segment of length $i+1$ of the vector $\vec{\beta}_i$.

Let us note that, according to Lemma 2, a periodic semimatrix is completely determined by specifying its first $q+r$ rows and the vectors $\vec{\alpha}_i, \vec{\beta}_i, \vec{\gamma}_i$.

We shall consider vectors of the form $\vec{\varphi} = ((\varphi_0, \ldots, \varphi_{r-1})$ over $S(X) = \langle\varepsilon, \sigma\rangle$, and shall call such a vector even if an even number of its components are equal to σ, and odd otherwise; the set of r-dimensional even vectors will be denoted by O_r, and of odd vectors by I_r.

Theorem 3. *If a finite-automaton mapping α in the binary alphabet $X = \langle x, y\rangle$ with canonical system of events $\langle W_x, W_y\rangle$ is induced by a periodic f.s. $(\alpha_{ij}(x))$, for which*

$$\vec{\alpha}_i = (\alpha_0^i(x), \ldots, \alpha_{r-1}^i(x)), \quad \vec{\beta}_i = (\beta_0^i(x), \ldots, \beta_{s-1}^i(x)), \quad \vec{\gamma}_i = (\gamma_0^i(y(x), \ldots, \gamma_{q-2}^i(x), \gamma_{q-1}^i)$$

are the vectors defining it by Lemma 2, then W_x is an event of the cyclic depth 2, which can be given by a regular expression having the following structure:

$$1)\quad W_x = P \vee Q; \qquad 2)\quad P = \bigvee_{j=0}^{q+r-1} P_j;$$

$$3)\quad P_j = \left(\bigvee_{\vec{\varphi}\in O_j} Q_{\varphi_0}^{\alpha_{j0}(x)} \cdots Q_{\varphi_{j-1}}^{\alpha_{j(j-1)}(x)}\right) P_x^{\alpha_{jj}} \;\vee\; \left(\bigvee_{\vec{\varphi}\in I_j} Q_{\varphi_0}^{\alpha_{j0}(x)} \cdots Q_{\varphi_{j-1}}^{\alpha_{j(j-1)}(x)}\right) P_y^{\alpha_{jj}} \;\vee$$

$$\left(\bigvee_{\vec{\varphi}\in O_{j-1}} Q_{\varphi_0}^{\alpha_{j0}(x)} \cdots Q_{\varphi_{j-2}}^{\alpha_{j(j-2)}(x)}\right) Q_{\xi}^{\alpha_{j(j-1)}(x)} X \;\vee$$

$$\left(\bigvee_{\vec{\varphi}\in I_{j-1}} Q_{\varphi_0}^{\alpha_{j0}(x)} \cdots Q_{\varphi_{j-2}}^{\alpha_{j(j-2)}(x)}\right) Q_{\eta}^{\alpha_{j(j-1)}(x)} X \;\vee$$

$$\left(\bigvee_{\vec{\varphi}\in O_{j-2}} Q_{\varphi_0}^{\alpha_{j0}(x)} \cdots Q_{\varphi_{j-3}}^{\alpha_{j(j-3)}(x)}\right) Q_{\xi}^{\alpha_{j(j-2)}(x)} X^2$$

$$\vee\; \left(\bigvee_{\vec{\varphi}\in I_{j-2}} Q_{\varphi_0}^{\alpha_{j0}(x)} \cdots Q_{\varphi_{j-3}}^{\alpha_{j(j-3)}(x)}\right) Q_{\eta}^{\alpha_{j(j-2)}(x)} X^2 \;\vee$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Q_{\varepsilon}^{\alpha_{j0}(x)} Q_{\xi}^{\alpha_{j1}(x)} X^{j-1} \;\vee\; Q_{\sigma}^{\alpha_{j0}(x)} Q_{\eta}^{\alpha_{j1}(x)} X^{j-1} \;\vee\; Q_{\xi}^{\alpha_{j0}(x)} X^{j};$$

4)

$$R = \bigvee_{i=0}^{s-1} R_i; \qquad 5)\; R_i = R_i^O \vee R_i^I;$$

5)

$$R_i = \Big[\Big(\bigvee_{\vec\varphi\,\vec\chi\in O_{r+i+q}} Q_{\vec\varphi}^{\vec a_i} S_i Q_{\chi_0}^{\delta_0^i(x)} \cdots Q_{\chi_{i+q-1}}^{\delta_{i+q-1}{}^{i}}\Big)\vee$$

$$\Big(\bigvee_{\vec\varphi\,\vec\chi\in I_{r+i+q}} Q_{\vec\varphi}^{\vec a_i} T_i Q_{\chi_0}^{\delta_0^i(x)} \cdots Q_{\chi_{i+q-1}}^{\delta_{i+q-1}{}^{i(x)}}\Big)\Big] P_x^{\delta_{i+q}{}^{i}} \vee$$

$$\Big(\bigvee_{j=1}^{i+q}\Big[\Big(\bigvee_{\vec\varphi\,\vec\chi\in O_{r+i+q-j}} Q_{\vec\varphi}^{\vec a_i} S_i Q_{\chi_0}^{\delta_0^i(x)} \cdots Q_{\chi_{i+q-j-1}}^{\delta_{i+q-j-1}{}^{i(x)}}\Big)\vee$$

$$\Big(\bigvee_{\vec\varphi\,\vec\chi\in I_{r+i+q-j}} Q_{\vec\varphi}^{\vec a_i} T_i Q_{\chi_0}^{\delta_0^i(x)} \cdots Q_{\chi_{i+q-j-1}}^{\delta_{i+q-j-1}{}^{i(x)}}\Big)\Big] Q_\xi^{\delta_{i+q-1}{}^{i(x)}} X^j\Big)\vee$$

$$\Big(\bigvee_{j=1}^{s}\Big[\Big(\bigvee_{\vec\varphi\,\vec\psi\in O_{r+s-j}} Q_{\vec\varphi}^{\vec a_i} S_i Q_{\psi_0}^{\beta_0^i(x)} \cdots Q_{\psi_{s-j-1}}^{\beta_{s-j-1}{}^{i}}\Big)\vee$$

$$\Big(\bigvee_{\vec\varphi\,\vec\psi\in I_{r+s-j}} Q_{\vec\varphi}^{\vec a_i} T_i Q_{\psi_0}^{\beta_0^i(x)} \cdots Q_{\psi_{s-j-1}}^{\beta_{s-j-1}{}^{i(x)}}\Big)\Big] Q_\xi^{\beta_{s-j}{}^{i(x)}} X^{j+i+q}\Big)\{X^s\}\vee$$

$$\Big(\bigvee_{j=1}^{r}\Big[\bigvee_{\vec\varphi\in O_{r-j}} Q_{\varphi_0}^{a_0^i(x)} \cdots Q_{\varphi_{r-j-1}}^{a_{r-j-1}{}^{i(x)}}\Big] Q_\xi^{a_{r-j}{}^{i(x)}} X^{j+i+q}\Big)\{X^s\},$$

where

$$(\delta_0^i(x), \ldots, \delta_i^i(x);\ \delta_{i+1}^i(x), \ldots, \delta_{i+q-1}^i(x), \delta_{i+q}^i) =$$

$$= (\beta_0^i(x), \ldots, \beta_i^i(x);\ \gamma_0^i(x), \ldots, \gamma_{q-2}^i(x), \gamma_{q-1}^i),$$

$$Q_{\vec\varphi}^{\vec a_i} = Q_{\varphi_0}^{a_0^i(x)} \cdots Q_{\varphi_{r-1}}^{a_{r-1}{}^{i}}; \qquad Q_{\psi_{-1}}^{\beta_{-1}{}^{j(x)}} = Q_{\chi_{-1}}^{\delta_{-1}{}^{i(x)}} = Q_{\varphi_{-1}}^{a_{-1}{}^{i(x)}} = \langle e\rangle$$

(e is the empty word);

7) R_i^I is obtained from R_i^O by replacing S_i by T_i and, conversely, x by y, ξ by η;

8)

$$S_i = \left\{\left(\bigvee_{\vec\psi\in O_s} Q_{\vec\psi}^{\vec\beta_i}\right)\vee\left(\bigvee_{\vec\psi\in I_s} Q_{\vec\psi}^{\vec\beta_i}\right)\left\{\bigvee_{\vec\psi\in O_s} Q_{\vec\psi}^{\vec\beta_i}\right\}\left(\bigvee_{\vec\psi\in I_s} Q_{\vec\psi}^{\vec\beta_i}\right)\right\},$$

$$T_i = \left\{\bigvee_{\vec\psi\in O_s} Q_{\vec\psi}^{\vec\beta_i}\right\}\left(\bigvee_{\vec\psi\in I_s} Q_{\vec\psi}^{\vec\beta_i}\right)\left\{\left(\bigvee_{\vec\psi\in O_s} Q_{\vec\psi}^{\vec\beta_i}\right)\vee\left(\bigvee_{\vec\psi\in I_s} Q_{\vec\psi}^{\vec\beta_i}\right)\left\{\bigvee_{\vec\psi\in O_s} Q_{\vec\chi}^{\vec\beta_i}\right\}\right\}\left(\bigvee_{\vec\psi\in I_s} Q_{\vec\psi}^{\vec\beta_i}\right),$$

where

$$Q_{\vec{\psi}}^{\vec{\beta}_i} = Q_{\psi_0}^{\beta_0^i(x)} \cdots Q_{\psi_{s-1}}^{{\beta_{s-1}}^{i(x)}}.$$

Conversely, if in the canonical system of events of a finite-automaton mapping the regular expression for the event W_x has the described structure under certain partitions $\alpha^i_{j_1}, \beta^i_{j_2}, \gamma^i_{j_3}$ of the alphabet X into the sets $Q_\zeta^{{\alpha_{j_1}}^i}, Q_\zeta^{{\beta_{j_2}}^i}, Q_\zeta^{{\gamma_{j_3}}^i}$ $(\zeta \in S'(X))$, respectively $(i = 0, 1, \dots, s-1;\ j_1 = 0, \dots, r-1;\ j_2 = 0, \dots, s-1;\ j_3 = 0, \dots, q-1)$, then from these partitions one reconstructs the functions $\alpha^i_{j_1}(x), \beta^i_{j_2}(x), \gamma^i_{j_3}(x)$, and from them, on the basis of Lemma 2, the functional semimatrix $(\alpha_{ij}(x))$ inducing α.

In the proof the following lemma is used, in particular.

Lemma 2. Let the events S, T, C, D in the alphabet X satisfy the relations

$$S = e \vee CS \vee DT; \qquad T = DS \vee CT,$$

where e is the empty word. Then

$$S = \{C \vee D\{C\}D\}; \qquad T = \{C\}D\{C \vee D\{C\}D\}.$$

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Note: Figure translations are in progress. See original paper for figures.